

2020 年全国硕士研究生入学统一考试数学 (三) 试题答案

1. 【答案】 B

$$\text{【解析】 } \lim_{x \rightarrow a} \frac{\sin f(x) - \sin a}{x - a} = \lim_{x \rightarrow a} \frac{\sin f(x) - \sin a}{f(x) - a} \cdot \frac{f(x) - a}{x - a} = b \cos a$$

2. 【答案】 A

【解析】 $f(x)$ 为奇函数, $f'(x)$ 为偶函数, 则

$\cos f(x)$ 为偶函数, $\cos f'(x)$ 为偶函数,

故 $\cos f(x) + f'(x)$ 为偶函数, 因此 $\int_0^x [\cos f'(t) + f(t)] dt$ 为奇函数

3. 【答案】 C

【解析】 间断点为 $x = -1, 0, 1, 2$,

$\lim_{x \rightarrow -1} f(x) = \infty$ 为无穷间断点, $\lim_{x \rightarrow 0} f(x) = -\frac{1}{2e}$ 为可去间断点,

$\lim_{x \rightarrow 1} f(x) = \infty$ 为无穷间断点, $\lim_{x \rightarrow 2} f(x) = \infty$ 为无穷间断点

4. 【答案】 B

【解析】 幂级数 $\sum_{n=1}^{\infty} na_n(x-2)^n$ 的收敛区间为 $(-2, 6)$, 则 $\sum_{n=1}^{\infty} na_n x^n$ 的收敛半径为 $R = 4$.

则 $\sum_{n=1}^{\infty} a_n x^{2n}$ 的收敛半径为 $R = 2$, 则 $\sum_{n=1}^{\infty} a_n(x+1)^{2n}$ 的收敛区间为 $(-3, 1)$.

5. 【答案】 C

【解析】 因为 A 不可逆, 所以 $r(A) < 4$, 又因为 $A_{12} \neq 0$, 所以 $r(A) \geq 3$,

故 $r(A) = 3, r(A^*) = 1$, 又因为 $A_{12} \neq 0$, 所以 $\alpha_1, \alpha_3, \alpha_4$ 线性无关,

又因为 $AA^* = O$, 所以

$A^*x = 0$ 的通解 $x = k_1\alpha_1 + k_2\alpha_3 + k_3\alpha_4$, 其中 k_1, k_2, k_3 为任意常数.

6. 【答案】 D

【解析】 由题知 $A\alpha_1 = \alpha_1, A\alpha_2 = \alpha_2, A\alpha_3 = -\alpha_3$,

所以 $A(\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2, A(-\alpha_3) = -(-\alpha_3)$,

$$\text{令 } P = (\alpha_1 + \alpha_2, -\alpha_3, \alpha_2), \text{ 则 } P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

7. 【答案】D

$$\begin{aligned} & \text{【解析】 } P(\overline{ABC}) + P(\overline{ABC}) + P(\overline{ABC}) \\ &= P(\overline{AB \cup C}) + P(\overline{BA \cup C}) + P(\overline{CA \cup B}) \\ &= P(A) - P(A(B \cup C)) + P(B) - P(B(A \cup C)) + P(C) - P(C(A \cup B)) \\ &= P(A) - P(AB \cup AC) + P(B) - P(AB \cup BC) + P(C) - P(AC \cup BC) \\ &= P(A) - P(AB) - P(AC) + P(B) - P(AB) - P(BC) + P(C) - P(AC) - P(BC) \\ &= \frac{5}{12}, \text{ 故选 } D. \end{aligned}$$

8. 【答案】C

$$\begin{aligned} & \text{【解析】 } X \sim N(0,1), Y \sim N(0,1), \rho_{xy} = -\frac{1}{2}, \text{ 则 } E(X+Y) = EX + EY = 0, \\ & D(X+Y) = DX + DY + 2Cov(X, Y) = 3, \text{ 从而 } \frac{X+Y}{\sqrt{3}} \sim N(0,1), \frac{X-Y}{\sqrt{7}} \sim N(0,1). \end{aligned}$$

9. 【答案】 $(\pi-1)dx-dy$

$$\text{【解析】 } dz = d \arctan(xy + \sin(x+y)) = \frac{ydx + xdy + \cos(x+y)(dx+dy)}{1+(xy+\sin(xy))^2},$$

$$\text{则 } dz \Big|_{(0,\pi)} = (\pi-1)dx - dy.$$

10. 【答案】 $y+1=x$

$$\begin{aligned} & \text{【解析】 两边关于 } x \text{ 求导 } 1 + y' + e^{2xy}(2y + 2xy') = 0 \\ & \text{在 } (0, -1) \text{ 处, } y' = 1 \text{ 切线方程为 } y+1=x. \end{aligned}$$

11. 【答案】 $Q=8$

$$\text{【解析】 } L(Q) = R(Q) - C(Q) = qp - C(Q)$$

当需求量与产量相同时最大

$$Q = \frac{800}{p+3} - 2$$

解得 $p = \frac{794-3Q}{Q+2}$, 所以 $L(Q) = Q \frac{794-3Q}{Q+2} - 100 - 13Q$

则 $L'(Q) = \frac{(794-6Q)(2+Q) - (794-3Q^2)}{(Q+2)^2} - 13$

令 $L'(Q) = 0$, 解得 $Q = 8 (Q > 0)$

而 $Q = 8$ 为唯一驻点, 则 $Q = 8$.

12. 【答案】 $\pi \ln 2 - \frac{\pi}{3}$

【解析】 $V = \int_0^1 2\pi x \frac{1}{1+x^2} dx - \int_0^1 2\pi x \frac{x}{2} dx = \pi \ln 2 - \frac{\pi}{3}$.

13. 【答案】 $a^4 - 4a^2$

【解析】
$$\begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{vmatrix} = a \begin{vmatrix} a & 1 & 0 \\ 1 & a & a \\ -1 & 0 & a \end{vmatrix} - \begin{vmatrix} 0 & a & 0 \\ -1 & 1 & a \\ 1 & -1 & a \end{vmatrix}$$
$$= -a \begin{vmatrix} 1 & a^2 \\ a & 2a \end{vmatrix} - 2a \begin{vmatrix} 0 & a \\ -1 & 1 \end{vmatrix}$$
$$= -a(2a - a^3) - 2a^2$$
$$= a^4 - 4a^2$$

14. 【答案】 $\frac{8}{7}$

【解析】 Y 可能的取值为 0, 1, 2.

当 $Y = 0$ 时, X 可能的取值为 3, 6, ..., $3n$, ..., 此时 $P(Y = 0) = \frac{1}{2^3} + \frac{1}{2^6} + \cdots + \frac{1}{2^{3n}} + \cdots$

当 $Y = 1$ 时, X 可能的取值为 1, 4, ..., $3n+1$, ..., 此时 $P(Y = 1) = \frac{1}{2} + \frac{1}{2^4} + \cdots + \frac{1}{2^{3n+1}} + \cdots$

当 $Y = 2$ 时, X 可能的取值为 2, 5, ..., $3n+2$, ..., 此时

$P(Y = 2) = \frac{1}{2^2} + \frac{1}{2^5} + \cdots + \frac{1}{2^{3n+2}} + \cdots$

则 $EY = 0 \cdot P(Y = 0) + 1 \cdot P(Y = 1) + 2 \cdot P(Y = 2) = 1 + \frac{1}{2^3} + \frac{1}{2^6} + \cdots + \frac{1}{2^{3n}} + \cdots$

$$= \lim_{n \rightarrow \infty} \frac{1 - \left(\frac{1}{2^3}\right)^n}{1 - \frac{1}{2^3}} = \frac{8}{7}.$$

$$15. \text{【解析】} 1 = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)^n - e}{\frac{b}{n^a}} = e \lim_{n \rightarrow \infty} \frac{e^{\frac{n \ln(1 + \frac{1}{n}) - 1}{n}} - 1}{\frac{b}{n^a}}$$

$$\text{令 } x = \frac{1}{n}, \text{ 则 } 1 = e \lim_{n \rightarrow \infty} \frac{e^{\frac{\ln(1+x)-1}{x}} - 1}{bx^a} = e \lim_{n \rightarrow \infty} \frac{\ln(1+x) - x}{bx^{a+1}} = e \lim_{n \rightarrow \infty} \frac{-\frac{1}{2}x^2}{bx^{a+1}}$$

$$\text{则 } \begin{cases} a+1=2 \\ -\frac{e}{2b}=1 \end{cases}, \text{ 从而 } \begin{cases} a=1 \\ b=-\frac{e}{2} \end{cases}.$$

$$16. \text{【解析】} \text{ 令 } \begin{cases} \frac{\partial f}{\partial x} = 3x^2 - y = 0, \\ \frac{\partial f}{\partial y} = 24y^2 - x = 0, \end{cases} \text{ 得驻点 } (0,0), \left(\frac{1}{6}, \frac{1}{12}\right)$$

$$\text{且 } \frac{\partial^2 f}{\partial x^2} = 6x, \frac{\partial^2 f}{\partial x \partial y} = -1, \frac{\partial^2 f}{\partial y^2} = 48y.$$

(x, y)	$A = \frac{\partial^2 f}{\partial x^2}$	$B = \frac{\partial^2 f}{\partial x \partial y}$	$C = \frac{\partial^2 f}{\partial y^2}$	$AC - B^2$	极值
$(0, 0)$	0	-1	0	< 0	无
$\left(\frac{1}{6}, \frac{1}{12}\right)$	1	-1	4	> 0	极小

$$\text{故 } f(x) \text{ 在 } \left(\frac{1}{6}, \frac{1}{12}\right) \text{ 处取得极小值且极小值 } f\left(\frac{1}{6}, \frac{1}{12}\right) = -\frac{1}{216}.$$

$$17. \text{【解析】} (1) \text{ 齐次方程的特征方程为 } r^2 + 2r + 5 = 0, \text{ 可得 } r_{1,2} = -1 \pm 2i,$$

$$f(x) = e^{-x}(C_1 \cos 2x + C_2 \sin 2x), \text{ 又 } f(0) = 1, f'(0) = -1, \text{ 解得 } C_1 = 1, C_2 = 0,$$

从而 $f(x) = e^{-x} \cos 2x$.

$$\begin{aligned}(2) \quad a_n &= \int_{n\pi}^{+\infty} f(x)dx = \int_{n\pi}^{+\infty} e^{-x} \cos 2x dx = \frac{1}{5}(-e^{-x} \cos 2x + 2e^{-x} \sin 2x) \Big|_{n\pi}^{+\infty} \\ &= \frac{1}{5} e^{-n\pi}\end{aligned}$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{5} e^{-n\pi} = \frac{1}{5} \lim_{n \rightarrow \infty} \frac{e^{-\pi}(1 - e^{-n\pi})}{1 - e^{-\pi}} = \frac{1}{5} \frac{e^{-\pi}}{1 - e^{-\pi}} = \frac{1}{5} \frac{1}{e^{\pi} - 1}$$

18. 【解析】 设 $\iint_D f(x, y) dx dy = A$

$$\text{由已知 } A = \iint_D y\sqrt{1-x^2} + xA dx dy$$

又 D 关于 y 轴对称, 则 $\iint_D xA dx dy = 0$

$$A = \iint_D y\sqrt{1-x^2} dx dy$$

$$= 2 \int_0^1 dx \int_0^{\sqrt{1-x^2}} y\sqrt{1-x^2} dy = \frac{3\pi}{16}$$

$$\iint_D xf(x, y) dx dy = \iint_D xy\sqrt{1-x^2} + Ax^2 dx dy$$

$$= \iint_D \frac{3\pi}{16} x^2 dx dy = \frac{3\pi^2}{128}.$$

19. 【解析】 (1) 若 $M = 0$, 则在 $[0, 2]$ 上 $f(x) \equiv 0$.

所以 $\forall \xi \in (0, 2)$, 有 $|f'(\xi)| = 0 \geq 0 = M$ 成立.

若 $M > 0$, 则设 $\exists c \in (0, 2)$ 使 $f(c) = M$

由拉格朗日中值定理, $\exists \xi_1 \in (0, c)$ 与 $\exists \xi_2 \in (c, 2)$, 使

$$f'(\xi_1) = \frac{f(c) - f(0)}{c - 0} = \frac{M}{c}, \quad f'(\xi_2) = \frac{f(2) - f(c)}{2 - c} = -\frac{M}{2 - c}$$

$$\text{所以 } |f'(\xi_1)| = \frac{M}{c}, \quad |f'(\xi_2)| = \frac{M}{2 - c}$$

若 $c \in (0, 1]$, 则可取 $\xi = \xi_1$, 有 $|f'(\xi)| = \frac{M}{c} \geq M$

若 $c \in (1, 2)$, 则可取 $\xi = \xi_2$, 有 $|f'(\xi)| = \frac{M}{2-c} \geq M$

综上所述, $\xi \in (0, 2)$, 使 $|f'(\xi)| \geq M$.

$$(2) \quad M = |f(x_0)| = |f(x_0) - f(0)| = \left| \int_0^{x_0} f'(x) dx \right| \leq \int_0^{x_0} |f'(x)| dx \leq \int_0^{x_0} M dx \leq M x_0$$

$$M = |f(x_0)| = |f(2) - f(x_0)| = \left| \int_{x_0}^2 f'(x) dx \right| \leq \int_{x_0}^2 |f'(x)| dx \leq \int_{x_0}^2 M dx \leq M(2 - x_0)$$

$$\text{故} \begin{cases} M(1-x_0) \leq 0 \\ M(1-x_0) \geq 0 \end{cases} \text{同时成立}$$

若 $x_0 \neq 1$ 则显然 $M = 0$

$$\text{若 } x_0 = 1 \text{ 则 } M \leq \int_0^1 |f'(x)| dx \leq \int_0^1 M dx = M \text{ 且 } M \leq \int_1^2 |f'(x)| dx \leq \int_1^2 M dx \leq M$$

当且仅当区间 $(0, 1)$ 和 $(1, 2)$ 上 $|f'(x)| \equiv M$ 成立, 若假设 $M \neq 0$

要么 $f'(x) \equiv \pm M$ 与 $f(0) = f(2) = 0$ 矛盾

$$\text{要么 } f'(x) = \begin{cases} \pm M, 0 < x < 1 \\ \mp M, 1 < x < 2 \end{cases}, f(x) \text{ 在 } x=1 \text{ 处不可导, 与已知矛盾}$$

故 $M = 0$

$$20. \text{【解析】} \text{ 设 } A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, B = \begin{pmatrix} a & 2 \\ 2 & b \end{pmatrix}, \text{ 由题存在正交矩阵 } Q, \text{ 使得 } Q^T A Q = B,$$

$$\text{即 } A \text{ 与 } B \text{ 合同, 且 } A \text{ 与 } B \text{ 相似, 故 } \begin{cases} |A| = |B| \\ \text{tr}(A) = \text{tr}(B) \end{cases}, \text{ 从而 } \begin{cases} ab - 4 = 0 \\ a + b = 5 \end{cases},$$

又由于 $a \geq b$, 所以 $a = 4, b = 1$.

$$\text{令 } |A - \lambda E| = \begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = \lambda(\lambda-5) = 0, \text{ 得 } \lambda_1 = 0, \lambda_2 = 5.$$

当 $\lambda_1 = 0$ 时, 求解 $AX = 0$ 得 $\alpha_1 = (2, 1)^T$;

当 $\lambda_2 = 5$ 时, 求解 $(A - 5E)X = 0$ 得 $\alpha_2 = (-1, 2)^T$.

所以存在正交阵 $P_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$, 使得 $P_1^T A P_1 = \Lambda = \begin{pmatrix} 0 & \\ & 5 \end{pmatrix}$.

当 $\lambda_1 = 0$ 时, 求解 $BX = 0$ 得 $\beta_1 = (-1, 2)^T$;

当 $\lambda_2 = 5$ 时, 求解 $(B - 5E)X = 0$ 得 $\beta_2 = (-1, 2)^T$.

所以存在正交阵 $P_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$, 使得 $P_2^T A P_2 = \Lambda = \begin{pmatrix} 0 & \\ & 5 \end{pmatrix}$.

从而 $P_1^T A P_1 = \begin{pmatrix} 0 & \\ & 5 \end{pmatrix} = P_2^T A P_2$, $P_2 P_1^T A P_1 P_2^T = B$, 所以 $Q = P_1 P_2^T = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$.

21. 【解析】(1) 因为 α 为非零向量, 且不是 A 的特征向量, 所以 $A\alpha \neq \lambda\alpha$, λ 为任一实数,

所以 $P = (\alpha, A\alpha)$ 的 2 列向量不成比例

所以 $\alpha, A\alpha$ 线性无关, 从而 $r(P) = 2$, 所以 P 可逆.

(2) 由于 $AP = A(\alpha, A\alpha) = (A\alpha, A^2\alpha) = (A\alpha, 6\alpha - A\alpha) = (\alpha, A\alpha) \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$

所以 $AP = P \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$, 又因为 P 可逆, 所以 $P^{-1}AP = \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$, 所以 $B = \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$,

又 $|B - \lambda E| = 0$, 所以 $\lambda^2 + \lambda - 6 = 0$, 从而 $\lambda_1 = 2, \lambda_2 = -3$,

所以 B 的 2 个特征值互不相同, 从而 B 可对角化, 又 A 与 B 相似, 所以 A 可对角化.

22. 【解析】(1) (Z_1, Z_2) 可能的取值为 $(0, 0), (0, 1), (1, 0), (1, 1)$, 由于 (X, Y) 服从均匀分布,

所以 $P(Z_1 = 0, Z_2 = 1) = P(X - Y \leq 0, X + Y > 0) = \frac{1}{2}$

$P(Z_1 = 1, Z_2 = 0) = P(X - Y > 0, X + Y \leq 0) = 0$

$$P(Z_1=0, Z_2=0) = P(X-Y \leq 0, X+Y \leq 0) = \frac{1}{4}$$

$$P(Z_1=1, Z_2=1) = P(X-Y > 0, X+Y > 0) = \frac{1}{4}$$

故 (Z_1, Z_2) 的分布律为:

	Z_2	0	1
Z_1			
0		$\frac{1}{4}$	$\frac{1}{2}$
1		0	$\frac{1}{4}$

$$(2) \because Z_1 \sim \begin{pmatrix} 0 & 1 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} \therefore EZ_1 = \frac{1}{4}, EZ_1^2 = \frac{1}{4}, DZ_1 = EZ_1^2 - (EZ_1)^2 = \frac{3}{16},$$

$$\because Z_2 \sim \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} \therefore EZ_2 = \frac{3}{4}, EZ_2^2 = \frac{3}{4}, DZ_2 = EZ_2^2 - (EZ_2)^2 = \frac{3}{16},$$

$$\because Z_1 Z_2 \sim \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} \therefore E(Z_1 Z_2) = \frac{1}{4},$$

$$\therefore \rho_{Z_1 Z_2} = \frac{Cor(Z_1 Z_2)}{\sqrt{DZ_1} \sqrt{DZ_2}} = \frac{EZ_1 Z_2 - EZ_1 EZ_2}{\sqrt{DZ_1} \sqrt{DZ_2}} = \frac{1}{3}.$$

23. 【解析】(1) $P\{T > s\} = 1 - P(T \leq s) = 1 - F(s) = e^{-\left(\frac{s}{\theta}\right)^m}$

$$P\{T > s+t | T > s\} = \frac{P(T > s+t)}{P(T > s)} = \frac{1 - P(T \leq s+t)}{1 - P(T \leq s)} = \frac{e^{-\left(\frac{s+t}{\theta}\right)^m}}{e^{-\left(\frac{s}{\theta}\right)^m}} = e^{\left(\frac{s}{\theta}\right)^m - \left(\frac{s+t}{\theta}\right)^m}$$

令 T 的密度函数为 $f(t)$, 则 $f(t) = F'(t) = \begin{cases} \frac{m}{\theta} \left(\frac{t}{\theta}\right)^{m-1} e^{-\left(\frac{t}{\theta}\right)^m} & t \geq 0 \\ 0 & t < 0 \end{cases}$

$$\text{令 } L(\theta) = \left(\frac{m}{\theta}\right)^n \frac{\prod_{i=1}^n t_i^{m-1}}{\theta^{n(m-1)}} e^{\frac{1}{\theta^m} \sum_{i=1}^n t_i^m}$$

$$\ln L'(\theta) = n \ln m - n \ln \theta + \ln \prod_{i=1}^n t_i^{m-1} - n(m-1) \ln \theta - \frac{1}{\theta^m} \sum_{i=1}^n t_i^m$$

$$\ln L'(\theta) = -\frac{n}{\theta} - \frac{n(m-1)}{\theta} + \frac{\sum_{i=1}^n t_i^m}{\theta^{m+1}} - mn\theta^m + \frac{\sum_{i=1}^n t_i^m}{\theta^{m+1}}$$

$$\text{令 } \ln L'(\theta) = 0, \text{解得 } \theta = \left[\frac{\sum_{i=1}^n t_i^m}{n} \right]^{\frac{1}{m}}$$

$$\text{令 } \hat{\theta} = \left[\frac{\sum_{i=1}^n t_i^m}{n} \right]^{\frac{1}{m}}, \text{ 则 } \hat{\theta} \text{ 为 } \theta \text{ 的最大似然估计量。}$$