

2020 年全国硕士研究生入学统一考试数学 (二) 试题答案

1. 【答案】D

【解析】选项 A, $(\int_0^x (e^{t^2} - 1) dt) = e^{x^2} - 1 \sim x^2 (x \rightarrow 0^+)$

选项 B, $(\int_0^x \ln(1 + \sqrt{t^3} dt)) = \ln(1 + \sqrt{x^3}) \sim x^{\frac{3}{2}} (x \rightarrow 0^+)$

选项 C, $(\int_0^{\sin x} \sin t^2 dt) = \sin(\sin^2 x) \cos x \sim x^2 (x \rightarrow 0^+)$

选项 D, $(\int_0^{1-\cos x} \sqrt{\sin^3 t} dt) = \sqrt{\sin^3(1-\cos x)} \sin x \sim cx^4 (x \rightarrow 0^+)$

2. 【答案】C

【解析】间断点为 $x = -1, 0, 1, 2$,

$\lim_{x \rightarrow -1} f(x) = \infty$ 为无穷间断点, $\lim_{x \rightarrow 0} f(x) = -\frac{1}{2e}$ 为可去间断点,

$\lim_{x \rightarrow 1} f(x) = \infty$ 为无穷间断点, $\lim_{x \rightarrow 2} f(x) = \infty$ 为无穷间断点.

3. 【答案】A

【解析】 $\int_0^1 \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} dx = 2 \int_0^1 \arcsin \sqrt{x} d \arcsin \sqrt{x} = \left(\arcsin \sqrt{x} \right)^2 \Big|_0^1 = \left(\frac{\pi}{2} \right)^2 = \frac{\pi^2}{4}$

4. 【答案】A

【解析】 $f^{(n)}(x) = \ln^{(n)}(1-x)x^2 + C_n^1 \ln^{(n-1)}(1-x)2x + C_n^2 \ln^{(n-2)}(1-x)2$

$f^{(n)}(0) = C_n^2 \ln^{(n-2)}(1-x)2 \Big|_{x=0} = n(n-1)(-1)^{n-3}(-1)^n(n-3)! = -\frac{n!}{n-2}$

5. 【答案】B

【解析】 $\frac{\partial f}{\partial x} \Big|_{(0,0)} = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x-0} = \lim_{x \rightarrow 0} \frac{x-0}{x-0} = 1$, ①对;

$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} xy = 0$, 则 $\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y) = 0$, ③与④对;

$$\frac{\partial^2 f}{\partial x \partial y} \Big|_{(0,0)} = \lim_{y \rightarrow 0} \frac{f_x(0,y) - f_x(0,0)}{y - 0} = \lim_{y \rightarrow 0} \frac{f_x(0,y) - 1}{y} \neq 1, \text{ ②错.}$$

于是正确的个数为 3 个.

6. 【答案】 B

【解析】 因为 $f'(x) > f(x) > 0$, 所以 $\frac{f'(x)}{f(x)} > 1$, 所以 $\frac{f'(x)}{f(x)} - 1 > 0$,

记 $F(x) = e^{-x} f(x)$, 则

$$F'(x) > 0, F(0) = f(0), F(-1) = ef(-1),$$

因为 $F(x)$ 单调增, 所以 $F(0) > F(-1)$,

$$\text{即 } f(0) > ef(-1), \text{ 即 } \frac{f(0)}{f(-1)} > e.$$

7. 【答案】 C

【解析】 因为 A 不可逆, 所以 $r(A) < 4$,

又因为 $A_{12} \neq 0$, 所以 $r(A) \geq 3$, 所以 $r(A) = 3, r(A^*) = 1$,

又因为 $A_{12} \neq 0$, 所以 $\alpha_1, \alpha_3, \alpha_4$ 线性无关,

又因为 $AA^* = O$, 所以 $A^*x = 0$ 的通解

$$x = k_1\alpha_1 + k_2\alpha_3 + k_3\alpha_4, \text{ 其中 } k_1, k_2, k_3 \text{ 为任意常数.}$$

8. 【答案】 D

【解析】 由题知 $A(\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2, A(-\alpha_3) = -(-\alpha_3)$,

$$\text{令 } P = (\alpha_1 + \alpha_2, -\alpha_3, \alpha_2), \text{ 则 } P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

9. 【答案】 $-\sqrt{2}$

$$\text{【解析】 } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{t + \sqrt{t^2 + 1}} (1 + \frac{t}{\sqrt{t^2 + 1}})}{\frac{t}{\sqrt{t^2 + 1}}} = \frac{1}{t},$$

$$\frac{d^2 y}{dx^2} = \frac{d\frac{1}{t}}{dx} = \frac{d\frac{1}{t}}{dt} \cdot \frac{dt}{dx} = -\frac{1}{t^2} \cdot \frac{1}{\frac{t}{\sqrt{t^2+1}}} = -\frac{\sqrt{t^2+1}}{t^3}, \text{ 则 } \left. \frac{d^2 y}{dx^2} \right|_{t=1} = -\sqrt{2}.$$

10. 【答案】 $\frac{2}{9}(2\sqrt{2}-1)$

【解析】 交换积分次序得

$$\begin{aligned} \int_0^1 dy \int_{\sqrt{y}}^1 \sqrt{x^3+1} dx &= \int_0^1 dx \int_0^{x^2} \sqrt{x^3+1} dy = \int_0^1 x^2 \sqrt{x^3+1} dx \\ &= \frac{1}{3} \int_0^1 \sqrt{x^3+1} d(x^3+1) = \frac{2}{9}(2\sqrt{2}-1). \end{aligned}$$

11. 【答案】 $(\pi-1)dx-dy$

【解析】 $dz = d \arctan(xy + \sin(x+y)) = \frac{ydx + xdy + \cos(x+y)(dx+dy)}{1+(xy+\sin(xy))^2},$

则 $dz|_{(0,\pi)} = (\pi-1)dx-dy.$

12. 【答案】 $\frac{1}{3}\rho ga^3$

【解析】 水压力为 $F = \int_0^a \rho g(a-y) \cdot 2y dy = 2\rho g \int_0^a (a-y) \cdot y dy = \frac{1}{3}\rho ga^3.$

13. 【答案】 1

【解析】 $y''+2y'+y=0$ 的特征方程为 $r^2+2r+1=0$, 则 $r=-1$ 为二重根,

微分方程的通解为 $y = (C_1 + C_2 x)e^{-x}.$

由 $y(0)=0, y'(0)=1$ 得 $C_1=0, C_2=1$, 则

$$y = xe^{-x}, \int_0^{+\infty} y(x) dx = \int_0^{+\infty} xe^{-x} dx = 1.$$

14. 【答案】 a^4-4a^2

$$\begin{aligned}
 \text{【解析】} \quad & \begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{vmatrix} = a \begin{vmatrix} a & 1 & 0 \\ 1 & a & a \\ -1 & 0 & a \end{vmatrix} - \begin{vmatrix} 0 & a & 0 \\ -1 & 1 & a \\ 1 & -1 & a \end{vmatrix} \\
 & = -a \begin{vmatrix} 1 & a^2 \\ a & 2a \end{vmatrix} - 2a \begin{vmatrix} 0 & a \\ -1 & 1 \end{vmatrix} \\
 & = -a(2a - a^3) - 2a^2 \\
 & = a^4 - 4a^2
 \end{aligned}$$

15. 【解析】只考虑 $x > 0$ 的情形:

$$k = \lim_{x \rightarrow +\infty} \frac{y}{x} = \lim_{x \rightarrow +\infty} \frac{x^x}{(1+x)^x} = \lim_{x \rightarrow +\infty} \frac{1}{\left(1 + \frac{1}{x}\right)^x} = \frac{1}{e},$$

$$b = \lim_{x \rightarrow +\infty} \left(y - \frac{1}{e}x \right) = \lim_{x \rightarrow +\infty} \left[\frac{x^{1+x}}{(1+x)^x} - \frac{x}{e} \right] = \lim_{x \rightarrow +\infty} \frac{ex^{1+x} - x(1+x)^x}{e(1+x)^x}$$

$$= \frac{1}{e} \lim_{x \rightarrow +\infty} x \left[e \left(\frac{x}{1+x} \right)^x - 1 \right] = \frac{1}{e} \lim_{x \rightarrow +\infty} x \left(e \cdot e^{\frac{x \ln x}{1+x}} - 1 \right) = \frac{1}{e} \lim_{x \rightarrow +\infty} x \left(e^{1+x \ln \frac{x}{1+x}} - 1 \right)$$

$$= \frac{1}{e} \lim_{x \rightarrow +\infty} x \left(1 + x \ln \frac{x}{1+x} \right) = \frac{1}{e} \lim_{x \rightarrow +\infty} \frac{1 + x \ln \frac{1}{1+\frac{1}{x}}}{\frac{1}{x}} = \frac{1}{e} \lim_{x \rightarrow +\infty} \frac{1 + \frac{1}{t} \ln \frac{1}{1+t}}{t}$$

$$= \frac{1}{e} \lim_{t \rightarrow 0^+} \frac{t - \ln(1+t)}{t^2} = \frac{1}{2e}, \text{ 于是, 曲线的斜渐近线方程为 } y = \frac{1}{e}x + \frac{1}{2e}.$$

16. 【解析】当 $x \neq 0$ 时,

$$g(x) = \int_0^1 f(xt) dt \stackrel{u=xt}{=} \frac{1}{x} \int_0^x f(u) du, g'(x) = \frac{1}{x^2} \int_0^x f(u) du + \frac{f(x)}{x},$$

当 $x = 0$ 时,

$$g'(x) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{x} \int_0^x f(u) du - 0}{x} = \lim_{x \rightarrow 0} \frac{\int_0^x f(u) du}{x^2} = \lim_{x \rightarrow 0} \frac{f(x)}{2x} = \frac{1}{2},$$

所以
$$g'(x) = \begin{cases} -\frac{1}{x^2} \int_0^x f(u) du - \frac{f(x)}{x}, & x \neq 0, \\ \frac{1}{2}, & x = 0, \end{cases}$$

$$\begin{aligned} \lim_{x \rightarrow 0} g'(x) &= \lim_{x \rightarrow 0} \left[-\frac{1}{x^2} \int_0^1 f(u) du + \frac{f(x)}{x} \right] = \lim_{x \rightarrow 0} \frac{-1}{x^2} \int_0^1 f(u) du + \lim_{x \rightarrow 0} \frac{f(x)}{x} \\ &= \lim_{x \rightarrow 0} \frac{-f(x)}{2x} + 1 = \frac{1}{2} = g'(0). \end{aligned}$$

所以 $g'(x)$ 在 $x=0$ 处连续.

17. 【解析】令
$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 - y = 0, \\ \frac{\partial f}{\partial y} = 24y^2 - x = 0, \end{cases} \quad \text{得驻点 } (0,0), \left(\frac{1}{6}, \frac{1}{12}\right)$$

且
$$\frac{\partial^2 f}{\partial x^2} = 6x, \frac{\partial^2 f}{\partial x \partial y} = -1, \frac{\partial^2 f}{\partial y^2} = 48y.$$

(x, y)	$A = \frac{\partial^2 f}{\partial x^2}$	$B = \frac{\partial^2 f}{\partial x \partial y}$	$C = \frac{\partial^2 f}{\partial y^2}$	$AC - B^2$	极值
$(0, 0)$	0	-1	0	< 0	无
$\left(\frac{1}{6}, \frac{1}{12}\right)$	1	-1	4	> 0	极小

故 $f(x)$ 在 $\left(\frac{1}{6}, \frac{1}{12}\right)$ 处取得极小值且极小值 $f\left(\frac{1}{6}, \frac{1}{12}\right) = -\frac{1}{216}$.

18. 【解析】令 $x = \frac{1}{t}$, 得
$$2f\left(\frac{1}{t}\right) + \frac{1}{t^2} f(t) = \frac{\frac{1}{t^2} + \frac{2}{t}}{\sqrt{1 + \frac{1}{t^2}}} = \frac{\frac{1}{t} + 2}{\sqrt{1 + t^2}},$$

即
$$2t^2 f\left(\frac{1}{t}\right) + f(t) = \frac{2t^2 + t}{\sqrt{1 + t^2}}, \quad \text{即 } 2x^2 f\left(\frac{1}{x}\right) + f(x) = \frac{2x^2 + x}{\sqrt{1 + x^2}},$$

与 $2f(x) + x^2 f\left(\frac{1}{x}\right) = \frac{x^2 + 2x}{\sqrt{1+x^2}}$ 联立, 得 $f(x) = \frac{x}{\sqrt{1+x^2}}$,

由 $y = f(x) = \frac{1}{2}$, 得 $x = \frac{\sqrt{3}}{3}$, 由 $y = \frac{\sqrt{3}}{2}$, 得 $x = \sqrt{3}$.

$$\begin{aligned} \text{体积为 } V &= \pi \left(\frac{\sqrt{3}}{2} \right)^2 \sqrt{3} - \pi \left(\frac{1}{2} \right)^2 \sqrt{3} - \left[\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \pi \left(\frac{x}{\sqrt{1+x^2}} \right) dx - \pi \left(\frac{1}{2} \right)^2 \left(\sqrt{3} - \frac{\sqrt{3}}{3} \right) \right] \\ &= \frac{\sqrt{3}}{2} \pi - \pi \left(\frac{2\sqrt{3}}{3} - \frac{\pi}{6} \right) + \frac{\sqrt{3}}{6} \pi = \frac{\pi^2}{6}. \end{aligned}$$

19. 【解析】 令 $I = \iint_D \frac{\sqrt{x^2 + y^2}}{x} dx dy, = \int_0^{\frac{\pi}{4}} \frac{1}{\cos \theta} d \int_{\frac{1}{\cos \theta}}^{\frac{2}{\cos \theta}} r dr$

$$\begin{aligned} &= \int_0^{\frac{\pi}{4}} \frac{1}{\cos \theta} \left(\frac{r^2}{2} \right) \bigg|_{\frac{1}{\cos \theta}}^{\frac{2}{\cos \theta}} d\theta = \frac{3}{2} \int_0^{\frac{\pi}{4}} \frac{1}{\cos^3 \theta} d\theta \\ &= \frac{3}{2} \int_0^{\frac{\pi}{4}} \sec \theta d \tan \theta = \frac{3}{2} \sec \theta \tan \theta \bigg|_0^{\frac{\pi}{4}} - \frac{3}{2} \int_0^{\frac{\pi}{4}} \tan^2 \theta \sec \theta d\theta \\ &= \int_0^{\frac{\pi}{4}} d\theta \int_{\sec \theta}^{2 \sec \theta} r \sec \theta dr = \frac{3}{2} \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta, \\ &= \frac{3}{2} \sqrt{2} - \frac{3}{2} \int_0^{\frac{\pi}{4}} (\sec^2 \theta - 1) \sec \theta d\theta \\ &= \frac{3}{2} \sqrt{2} - \frac{3}{2} \int_0^{\frac{\pi}{4}} \sec^3 \theta + \frac{3}{2} \int_0^{\frac{\pi}{4}} \sec \theta d\theta \\ &= \frac{3}{2} \sqrt{2} - I + \frac{3}{2} \ln(\sec \theta + \tan \theta) \bigg|_0^{\frac{\pi}{4}} \\ &= \frac{3}{2} \sqrt{2} - I + \frac{3}{2} \ln(1 + \sqrt{2}) \end{aligned}$$

$$I = \frac{3}{4} [\sqrt{2} + \ln(1 + \sqrt{2})].$$

20. 证明: (1) 令 $F(x) = (2-x)f(x)$, 由题意 $f(1) = 0, F(1) = 0, F(2) = 0$,
因为 $F(x)$ 在 $[1, 2]$ 上连续, 在 $(1, 2)$ 可导,
所以由罗尔定理可知 $\exists \xi \in (1, 2)$ 使 $F'(\xi) = 0$,

$$\text{即 } f(\xi) = (2 - \xi)e^{\xi^2}$$

(2) 令 $g(x) = \ln x, f(x), g(x)$ 在 $[1, 2]$ 上连续, 在 $(1, 2)$ 可导, 且 $g'(x) \neq 0$,

所以由柯西中值定理可知

$$\text{存在 } \eta \in (1, 2), \text{ 使得 } \frac{f'(\eta)}{g'(\eta)} = \frac{f(2) - f(1)}{g(2) - g(1)}, \text{ 即 } f(2) = \ln 2 \cdot \eta e^{\eta^2}.$$

21. 【解析】 设所求曲线方程为 $y = y(x)$, 任一点 M 坐标为 (x, y) ,

$$\text{由题意得 } \tan \theta = y' = \frac{MP}{TP}, \text{ 即 } TP = \frac{y}{y'},$$

三角形 MTP 的面积为:

$$S = \frac{1}{2} MP \times TP = \frac{1}{2} \times y \times \frac{y}{y'} = \frac{y^2}{2y'},$$

$$\text{曲边三角形 } OMP \text{ 的面积 } S = \int_0^x y(x) dx,$$

$$\text{由两面积之比为常数得 } \frac{y^2}{2y'} = \frac{2}{3} \int_0^x y(x) dx,$$

$$\text{两边关于 } x \text{ 求导得 } \frac{2yy'y' - y^2y''}{y'^2} = \frac{4}{3} y(x), \text{ 即 } y'y'' = \frac{2}{3} y'^2,$$

$$\text{令 } p(y) = y', \text{ 则 } y'' = p \frac{dp}{dy},$$

$$\text{原方程化为 } yp \frac{dp}{dy} = \frac{2}{3} p^2, \text{ 即 } p \left[y \frac{dp}{dy} - \frac{2}{3} p \right] = 0.$$

$$\text{由 } y \frac{dp}{dy} - \frac{2}{3} p = 0, \text{ 得 } p = C_1 y^{\frac{2}{3}}, \text{ 即 } y' = C_1 y^{\frac{2}{3}}, \text{ 从而 } C_1 + C_2 = \frac{y^{\frac{1}{3}}}{\frac{1}{3}},$$

由曲线过原点, 得 $y|_{x=0} = 0$, 代入得 $C_2 = 0$.

所求曲线为 $y^{\frac{1}{3}} = \frac{1}{3}C_1x$,

由 C_1 的任意性, 曲线可表示为 $y = Cx^3$, C 为任意常数。

22. 【解析】(1) 设 $A = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, 由题意可知

$r(A) = r(B)$, 而 $r(B) = 2$, 故 $r(A) = 2$, 于是可得 $a = -\frac{1}{2}$.

(3) 对于二次型 f

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_1x_3$$

$$= \left(x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3\right)^2 + \frac{3}{4}x_2^2 + \frac{3}{4}x_3^2 - \frac{3}{2}x_2x_3$$

$$= \left(x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3\right)^2 + \frac{3}{4}(x_2 - x_3)^2$$

$$\text{令} \begin{cases} z_1 = x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ z_2 = \frac{\sqrt{3}}{2}(x_2 - x_3) \\ z_3 = x_3 \end{cases}, \text{即} \begin{cases} x_1 = z_1 + \frac{1}{\sqrt{3}}z_2 + z_3 \\ x_2 = \frac{2}{\sqrt{3}}z_2 + z_3 \\ x_3 = z_3 \end{cases}, \text{得 } f = z_1^2 + z_2^2,$$

$$\text{取 } P_1 = \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & 1 \\ 0 & \frac{2}{\sqrt{3}} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

对于二次型 g , $g(y_1, y_2, y_3) = y_1^2 + y_2^2 + 4y_3^2 + 2y_1y_2 = (y_1 + y_2)^2 + 4y_3^2$,

$$\begin{pmatrix} z_1 = y_1 + y_2 \\ z_2 = 2y_3 \\ z_3 = y_2 \end{pmatrix}, \text{即} \begin{pmatrix} y_1 = z_1 - z_3 \\ y_2 = z_3 \\ y_3 = \frac{1}{2}z_2 \end{pmatrix}, \text{得} g = z_1^2 + z_2^2, \text{取} P_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$\text{取 } P = P_1 P_2^{-1} = \begin{pmatrix} 1 & 2 & \frac{2}{\sqrt{3}} \\ 0 & 1 & \frac{4}{\sqrt{3}} \\ 0 & 1 & 0 \end{pmatrix}, \text{存在变换} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = P \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \text{使得 } f(x_1, x_2, x_3) \text{ 化为 } g(y_1, y_2, y_3).$$

23.证明: (1) 因为 a 是非 0 向量, 且不是 A 的特征向量,

所以 $Aa \neq \lambda a, \lambda$ 为任意实数,

所以, $P = (a, Aa)$ 的 2 列向量不成比例,

所以 a, Aa 线性无关, 从而 $R(P) = 2$, 所以 P 可逆。

$$(2) \text{ 由于 } AP = A(a, Aa) = (Aa, A^2a) = (Aa, 6a - Aa) = (a, Aa) \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$$

$$\text{所以 } AP = P \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix},$$

$$\text{又因为 } P \text{ 可逆, 所以 } P^{-1}AP = \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix},$$

$$\text{所以, } B = \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}, \text{又 } |B - \lambda E| = 0, \text{所以, } \lambda^2 + \lambda - 6 = 0$$

$$\text{从而 } \lambda_1 = -3, \lambda_2 = 2$$

所以 B 的 2 个特征值互不相同, 从而 B 可对角化

又 A 与 B 相似, 所以 A 可对角化。