

2018年（数一）真题答案解析

一、选择题

(1) D

解 对于 D 选项 $f(x) = \cos \sqrt{|x|}$,

$$\text{由 } f'_+(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^+} \frac{\cos \sqrt{|x|} - 1}{x} = \frac{-\frac{1}{2}x}{x} = -\frac{1}{2},$$

$$f'_-(0) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^-} \frac{\cos \sqrt{|x|} - 1}{x} = \frac{\frac{1}{2}x}{x} = \frac{1}{2},$$

可得 $f'_+(0) \neq f'_-(0)$, 因此 $f(x)$ 在 $x=0$ 处不可导. 故应选 D.

(2) B

解 已知平面过点 $(1, 0, 0)$, $(0, 1, 0)$ 两点, 可得同平面内一向量 $(1, -1, 0)$, 曲面 $z = x^2 + y^2$ 的切平面法向量为 $(2x, 2y, -1)$. 所以 $2x - 2y = 0$, 即 $x = y$. 故应选 B.

(3) B

$$\text{解 原式} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n \frac{2}{(2n+1)!},$$

$$\text{易知 } \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^n = \cos x, \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n+1)!} = \sin x.$$

则原式 $= 2\sin 1 + \cos 1$. 故应选 B.

(4) C

$$\text{解 利用对称性可计算 } M = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+x)^2}{1+x^2} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + \frac{2x}{1+x^2}\right) dx = \pi.$$

易得, $K > \pi$, $N < \pi$. 所以 $K > M > N$. 故应选 C.

(5) A

解 易知题中矩阵的特征值均为 3 重特征值 1, 若矩阵相似, 则特征值对应的 $\lambda E - A$,

$$\text{即 } E - A \text{ 秩必然相等, 显然 } E - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ 的秩为 2.}$$

故应选 A.

(6) A

解 对于 B 选项, 若 $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, 则 $r(A \quad BA) = 2 \neq r(A)$, 排除 B.

对于 C 选项, 若 $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, 则 $r(A \quad B) = 2 \neq \max\{r(A), r(B)\}$, 排除 C.

对于 D 选项, 若 $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, 则 $r(A \quad B) = 2 \neq r(A^T \quad B^T)$, 排除 D.

故应选 A.

(7) A

解 由 $f(1+x) = f(1-x)$ 可知, $f(x)$ 关于 $x = 1$ 对称, 所以 $\int_{-\infty}^1 f(x) dx = \int_1^{+\infty} f(x) dx = 0.5$.

又已知, $\int_0^2 f(x) dx = 0.6$, 则 $\int_0^1 f(x) dx = \int_1^2 f(x) dx = 0.3$.

所以, $P\{X < 0\} = \int_{-\infty}^0 f(x) dx = \int_{-\infty}^1 f(x) dx - \int_0^1 f(x) dx = 0.2$.

故应选 A.

(8) D

解 若显著性水平 $\alpha = 0.05$ 时可接受 H_0 , 则检验统计量 $|Z| \leq U_{0.025}$, 则 $|Z| \leq U_{0.005}$.

故应选 D.

二、填空题

(9) -2

解 原式 $= \lim_{x \rightarrow 0} \frac{\left(\frac{1-\tan x}{1+\tan x} - 1\right)}{\sin kx} = e$, 则 $\lim_{x \rightarrow 0} \frac{\left(\frac{1-\tan x}{1+\tan x} - 1\right)}{\sin kx} = 1$.

即 $\lim_{x \rightarrow 0} \frac{-2\tan x}{(1+\tan x)\sin kx} = \frac{-2x}{kx} = 1$, 所以 $k = -2$.

故应填 -2.

(10) $2(\ln 2 - 1)$

解 $y = f(x)$ 过点 $(0, 0)$, 即 $f(x) = 0$, $y = f(x)$ 与 $y = 2^x$ 在点 $(1, 2)$ 相切 $\Rightarrow f(1) = 2$ 且 $f'(1) = 2\ln 2$.

$\int_0^1 x f''(x) dx = x f'(x) \Big|_0^1 - \int_0^1 f'(x) dx = f'(1) - (f(1) - f(0)) = 2\ln 2 - 2 = 2(\ln 2 - 1)$.

故应填 $2(\ln 2 - 1)$.

(11) $i - k$

解 $\text{rot} \mathbf{F} = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -yz & xz \end{pmatrix} = (y, -z, -x) \mid (1, 1, 0) = (1, 0, -1) = i - k$.

故应填 $i - k$.

(12) $-\frac{\pi}{3}$

解 $L = \begin{cases} x^2 + y^2 + z^2 = 1, \\ x + y + z = 0, \end{cases}$ 则 $\oint_L xy ds = \oint_L \left| \frac{1}{2} - (x^2 + y^2) \right| ds = \oint_L \left(\frac{1}{2} - \frac{2}{3} \right) ds = -\frac{\pi}{3}$.

故应填 $-\frac{\pi}{3}$.

(13) -1

解 设 A 特征值为 λ_1, λ_2 , 对应的特征向量分别为 α_1, α_2 , 则 $A\alpha_1 = \lambda_1\alpha_1, A\alpha_2 = \lambda_2\alpha_2$,

$A(\alpha_1 + \alpha_2) = \lambda_1\alpha_1 + \lambda_2\alpha_2$.

$A^2(\alpha_1 + \alpha_2) = A(\lambda_1\alpha_1 + \lambda_2\alpha_2) = \lambda_1^2\alpha_1 + \lambda_2^2\alpha_2 = \alpha_1 + \alpha_2$, 则 $\lambda_1 = \pm 1, \lambda_2 = \pm 1$,

又因为 $\lambda_1 \neq \lambda_2$, 所以 $|A| = \lambda_1 \lambda_2 = -1$.

故应填 -1 .

$$(14) \frac{1}{4}$$

解 $P(AC | AB \cup C) \cdot P(AB \cup C) = P(AC)$

$$\frac{1}{4} \cdot \left[\frac{1}{4} + P(C) \right] = \frac{1}{2} \cdot P(C)$$

解得 $P(C) = \frac{1}{4}$, 故应填 $\frac{1}{4}$.

三、解答题

$$\begin{aligned} (15) \text{ 解 } \int e^{2x} \arctan \sqrt{e^x - 1} dx &= \frac{1}{2} \int \arctan \sqrt{e^x - 1} de^{2x} \\ &= \frac{1}{2} e^{2x} \arctan \sqrt{e^x - 1} - \frac{1}{4} \int \frac{e^{2x}}{\sqrt{e^x - 1}} dx. \end{aligned}$$

$$\begin{aligned} \text{又 } \int \frac{e^{2x}}{\sqrt{e^x - 1}} dx &= \int \frac{e^x}{\sqrt{e^x - 1}} de^x \\ &= \int \sqrt{e^x - 1} de^x + \int \frac{1}{\sqrt{e^x - 1}} de^x \\ &= \frac{2}{3} (e^x - 1) \sqrt{e^x - 1} + 2 \sqrt{e^x - 1} + C, \end{aligned}$$

$$\text{所以 } \int e^{2x} \arctan \sqrt{e^x - 1} dx = \frac{1}{2} e^{2x} \arctan \sqrt{e^x - 1} - \frac{1}{6} (e^x + 2) \sqrt{e^x - 1} + C.$$

(16) 解 设圆的半径为 x , 正方形与正三角形的边长分别为 y 和 z , 则问题化为: 函数

$f(x, y, z) = \pi x^2 + y^2 + \frac{\sqrt{3}}{4} z^2$ 在条件 $2\pi x + 4y + 3z = 2$ ($x > 0, y > 0, z > 0$) 下是否存在最小值.

令 $L(x, y, z, \lambda) = \pi x^2 + y^2 + \frac{\sqrt{3}}{4} z^2 + \lambda(2\pi x + 4y + 3z - 2)$, 考虑方程组

$$\begin{cases} \frac{\partial L}{\partial x} = 2\pi x + 2\pi\lambda = 0, \\ \frac{\partial L}{\partial y} = 2y + 4\lambda = 0, \\ \frac{\partial L}{\partial z} = \frac{\sqrt{3}}{2} z + 3\lambda = 0, \\ \frac{\partial L}{\partial \lambda} = 2\pi x + 4y + 3z - 2 = 0, \end{cases}$$

$$\text{解得 } x_0 = \frac{1}{\pi + 4 + 3\sqrt{3}}, \quad y_0 = \frac{2}{\pi + 4 + 3\sqrt{3}}, \quad z_0 = \frac{2\sqrt{3}}{\pi + 4 + 3\sqrt{3}}.$$

$$f(x_0, y_0, z_0) = \frac{1}{\pi + 4 + 3\sqrt{3}}.$$

又当 $2\pi x + 4y + 3z = 2$ 且 $xyz = 0$ 时, $f(x, y, z)$ 的最小值为

$$f\left(0, \frac{2}{4+3\sqrt{3}}, \frac{2\sqrt{3}}{4+3\sqrt{3}}\right) = \frac{1}{4+3\sqrt{3}},$$

所以三个图形的面积之和存在最小值, 最小值为

$$f(x_0, y_0, z_0) = \frac{1}{\pi + 4 + 3\sqrt{3}} (\text{单位: m}^2).$$

(17) 解 设 Σ_1 为平面 $x=0$ 被 $\begin{cases} 3y^2 + 3z^2 = 1, \\ x=0, \end{cases}$ 所围部分的后侧, Ω 为 Σ 与 Σ_1 所围的立体.

根据高斯公式,

$$\iint_{\Sigma+\Sigma_1} x \, dy \, dz + (y^3 + 2) \, dz \, dx + z^3 \, dx \, dy = \iiint_{\Omega} (1 + 3y^2 + 3z^2) \, dx \, dy \, dz.$$

设 $y = r \cos \theta$, $z = r \sin \theta$, 则

$$\begin{aligned} \iiint_{\Omega} (1 + 3y^2 + 3z^2) \, dx \, dy \, dz &= \int_0^{2\pi} d\theta \int_0^{\frac{\sqrt{3}}{3}} dr \int_0^{\sqrt{1-3r^2}} (1 + 3r^2) r \, dx \\ &= 2\pi \int_0^{\frac{\sqrt{3}}{3}} r(1 + 3r^2) \sqrt{1-3r^2} \, dr. \end{aligned}$$

设 $\sqrt{1-3r^2} = t$, 则

$$\begin{aligned} 2\pi \int_0^{\frac{\sqrt{3}}{3}} r(1 + 3r^2) \sqrt{1-3r^2} \, dr &= \frac{2\pi}{3} \int_0^1 (2-t^2) t^2 \, dt \\ &= \frac{14\pi}{45}. \end{aligned}$$

又 $\iint_{\Sigma_1} x \, dy \, dz + (y^3 + 2) \, dz \, dx + z^3 \, dx \, dy = 0$, 所以 $I = \frac{14\pi}{45}$.

(18) 解 (I) 当 $f(x) = x$ 时, 方程化为 $y' + y = x$, 其通解为

$$\begin{aligned} y &= e^{-x} \left(C + \int x e^x \, dx \right) \\ &= e^{-x} (C + x e^x - e^x) \\ &= C e^{-x} + x - 1. \end{aligned}$$

(II) 方程 $y' + y = f(x)$ 的通解为

$$y = e^{-\int_0^x dt} \left(C + \int_0^x e^{\int_0^s ds} f(t) \, dt \right),$$

$$\text{即 } y = e^{-x} \left(C + \int_0^x e^t f(t) \, dt \right).$$

由 $y(x) = e^{-x} \left(C + \int_0^x e^t f(t) \, dt \right)$, 得

$$y(x+T) - y(x) = e^{-x} \left[\left(\frac{1}{e^T} - 1 \right) C + \frac{1}{e^T} \int_0^{x+T} e^t f(t) \, dt - \int_0^x e^t f(t) \, dt \right].$$

因为 $f(x)$ 是周期为 T 的连续函数, 所以

$$\frac{1}{e^T} \int_0^{x+T} e^t f(t) \, dt = \frac{1}{e^T} \int_0^T e^t f(t) \, dt + \frac{1}{e^T} \int_T^{x+T} e^t f(t) \, dt$$

$$\begin{aligned}
&= \frac{1}{e^T} \int_0^T e^t f(t) dt + \frac{1}{e^T} \int_0^x e^{u+T} f(u+T) du \\
&= \frac{1}{e^T} \int_0^T e^t f(t) dt + \int_0^x e^t f(t) dt.
\end{aligned}$$

从而 $y(x+T) - y(x) = e^{-x} \left[\left(\frac{1}{e^T} - 1 \right) C + \frac{1}{e^T} \int_0^T e^t f(t) dt \right]$.

所以,当且仅当 $C = \frac{1}{e^T - 1} \int_0^T e^t f(t) dt$ 时, $y(x+T) - y(x) = 0$.

故方程存在唯一的以 T 为周期的解.

(19) 解 由于 $x_1 \neq 0$, 所以 $e^{x_2} = \frac{e^{x_1} - 1}{x_1}$.

根据微分中值定理, 存在 $\xi \in (0, x_1)$, 使得 $\frac{e^{x_1} - 1}{x_1} = e^\xi$.

所以 $e^{x_2} = e^\xi$, 故 $0 < x_2 < x_1$.

假设 $0 < x_{n+1} < x_n$, 则

$$e^{x_{n+2}} = \frac{e^{x_{n+1}} - 1}{x_{n+1}} = e^\eta \quad (0 < \eta < x_{n+1}),$$

所以 $0 < x_{n+2} < x_{n+1}$.

故 $\{x_n\}$ 是单调减少的数列, 且下有界, 从而 $\{x_n\}$ 收敛.

设 $\lim_{n \rightarrow \infty} x_n = a$, 得 $a e^a = e^a - 1$. 易知 $a = 0$ 为其解.

令 $f(x) = x e^x - e^x + 1$, 则 $f'(x) = x e^x$.

当 $x > 0$ 时, $f'(x) > 0$, 函数 $f(x)$ 在 $[0, +\infty)$ 上单调增加, 所以 $a = 0$ 是方程 $a e^a = e^a - 1$ 在 $[0, +\infty)$ 上的唯一的解, 故 $\lim_{n \rightarrow \infty} x_n = 0$.

(20) 解 (I) $f(x_1, x_2, x_3) = 0$ 当且仅当

$$\begin{cases} x_1 - x_2 + x_3 = 0, \\ x_2 + x_3 = 0, \\ x_1 + a x_3 = 0, \end{cases}$$

对方程组的系数矩阵施以初等行变换得

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & a \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & a-2 \end{pmatrix}$$

当 $a \neq 2$ 时, 方程组只有零解, 故 $f(x_1, x_2, x_3) = 0$ 的解为 $\mathbf{x} = \mathbf{0}$,

当 $a = 2$ 时, 方程组有无穷多解, 通解为 $\mathbf{x} = k \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$, k 为任意常数,

故 $f(x_1, x_2, x_3) = 0$ 的解是 $\mathbf{x} = k \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$, k 为任意常数.

(II) 由(I)知, 当 $a \neq 2$ 时, $f(x_1, x_2, x_3)$ 正定, $f(x_1, x_2, x_3)$ 的规范形为 $y_1^2 + y_2^2 + y_3^2$.

当 $a = 2$ 时,

$$\begin{aligned} f(x_1, x_2, x_3) &= 2x_1^2 + 2x_2^2 + 6x_3^2 - 2x_1x_2 + 6x_1x_3 \\ &= 2\left(x_1 - \frac{1}{2}x_2 + \frac{3}{2}x_3\right)^2 + \frac{3}{2}(x_2 + x_3)^2, \end{aligned}$$

所以 $f(x_1, x_2, x_3)$ 的规范形为 $y_1^2 + y_2^2$.

(21) 解 (I) 对矩阵 $\mathbf{A}, \mathbf{B}$ 分别施以初等行变换得

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 1 & 2 & a \\ 1 & 3 & 0 \\ 2 & 7 & -a \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3a \\ 0 & 1 & -a \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{B} &= \begin{pmatrix} 1 & a & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2-a \end{pmatrix}. \end{aligned}$$

由题设知 $a = 2$.

(II) 由(I)知 $a = 2$, 对矩阵 $(\mathbf{A} : \mathbf{B})$ 施以初等行变换得

$$(\mathbf{A} : \mathbf{B}) = \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 2 & 2 \\ 1 & 3 & 0 & 0 & 1 & 1 \\ 2 & 7 & -2 & -1 & 1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 6 & 3 & 4 & 4 \\ 0 & 1 & -2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

记 $\mathbf{B} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3)$, 由于

$$\mathbf{A} \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix} = \mathbf{0}, \mathbf{A} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \boldsymbol{\beta}_1, \mathbf{A} \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} = \boldsymbol{\beta}_2, \mathbf{A} \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} = \boldsymbol{\beta}_3,$$

故 $\mathbf{AX} = \mathbf{B}$ 的解为

$$\mathbf{X} = \begin{pmatrix} 3-6k_1 & 4-6k_2 & 4-6k_3 \\ -1+2k_1 & -1+2k_2 & -1+2k_3 \\ k_1 & k_2 & k_3 \end{pmatrix}, \text{ 其中 } k_1, k_2, k_3 \text{ 为任意常数.}$$

由于 $|\mathbf{X}| = k_3 - k_2$, 所以满足 $\mathbf{AP} = \mathbf{B}$ 的可逆矩阵为

$$\mathbf{P} = \begin{pmatrix} 3-6k_1 & 4-6k_2 & 4-6k_3 \\ -1+2k_1 & -1+2k_2 & -1+2k_3 \\ k_1 & k_2 & k_3 \end{pmatrix}, \text{ 其中 } k_2 \neq k_3.$$

(22) 解 (I) 由题设可得

$$EX = (-1) \times \frac{1}{2} + 1 \times \frac{1}{2} = 0,$$

$$E(XZ) = E(X^2Y) = EX^2 \cdot EY = \lambda.$$

$$\text{所以 } \text{Cov}(X, Z) = E(XZ) - EX \cdot EZ = \lambda.$$

(II) Z 的所有可能取值为全体整数值, 且

$$P\{Z=0\} = P\{Y=0\} = e^{-\lambda};$$

对于 $n = \pm 1, \pm 2, \dots$, 有

$$P\{Z=n\} = P\{XY=n\}$$

$$\begin{aligned}
&= P\left\{X = \frac{n}{|n|}, Y = |n|\right\} \\
&= P\left\{X = \frac{n}{|n|}\right\} P\{Y = |n|\} \\
&= e^{-\lambda} \frac{\lambda^{|n|}}{2 \cdot |n|!}.
\end{aligned}$$

(23) 解 (I) 设 $x_1, x_2, \dots, x_n$ 为样本观测值, 似然函数为

$$L(\sigma) = \prod_{i=1}^n f(x_i; \sigma) = \frac{1}{2^n \sigma^n} e^{-\frac{1}{\sigma} \sum_{i=1}^n |x_i|},$$

$$\text{则 } \ln L(\sigma) = -n \ln 2 - n \ln \sigma - \frac{1}{\sigma} \sum_{i=1}^n |x_i|.$$

$$\text{令 } \frac{d \ln L(\sigma)}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^n |x_i| = 0, \text{ 解得}$$

$$\sigma = \frac{1}{n} \sum_{i=1}^n |x_i|.$$

$$\text{所以 } \hat{\sigma} = \frac{1}{n} \sum_{i=1}^n |X_i|.$$

$$(II) \text{ 由于 } E|X| = \int_{-\infty}^{+\infty} |x| f(x; \sigma) dx = \int_{-\infty}^{+\infty} |x| \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}} dx = \frac{1}{\sigma} \int_0^{+\infty} x e^{-\frac{x}{\sigma}} dx = \sigma, \text{ 所以}$$

$$E\hat{\sigma} = \frac{1}{n} \sum_{i=1}^n E|X_i| = E|X| = \sigma, \quad .$$

又因为

$$E|X|^2 = EX^2 = \int_{-\infty}^{+\infty} x^2 f(x; \sigma) dx = \int_{-\infty}^{+\infty} x^2 \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}} dx = \frac{1}{\sigma} \int_0^{+\infty} x^2 e^{-\frac{x}{\sigma}} dx = 2\sigma^2,$$

$$D(|X|) = E(|X|^2) - (E|X|)^2 = \sigma^2,$$

所以

$$D\hat{\sigma} = \frac{1}{n^2} \sum_{i=1}^n D(|X_i|) = \frac{D(|X|)}{n} = \frac{\sigma^2}{n}.$$