

2014年(数一) 真题答案解析

一、选择题

(1) C

解 由渐近线定义可知,四个选项的曲线均不存在水平渐近线和垂直渐近线.

对于 $y = x + \sin \frac{1}{x}$, 可知

$$a = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{x + \sin \frac{1}{x}}{x} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \sin \frac{1}{x} \right) = 1,$$

$$b = \lim_{x \rightarrow \infty} [f(x) - ax] = \lim_{x \rightarrow \infty} \sin \frac{1}{x} = 0.$$

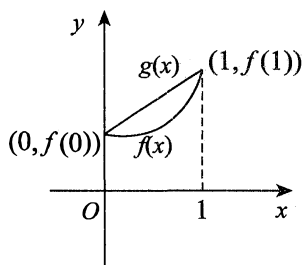
所以 $y = x$ 是 $y = x + \sin \frac{1}{x}$ 的斜渐近线. 故应选 C.

(2) D

解 当 $f''(x) \geq 0$ 时, $f(x)$ 是凹函数.

而 $g(x) = [f(1) - f(0)]x + f(0)$ 可视为连接 $(0, f(0))$ 与 $(1, f(1))$ 的直线段, 如右图所示, 则 $f(x) \leq g(x)$.

故应选 D.

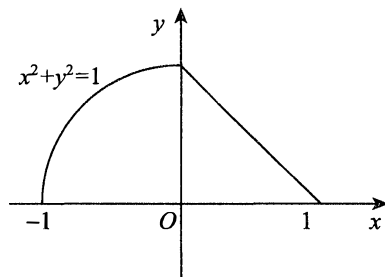


(3) D

解 积分区域如右图所示, 换成极坐标则为

$$\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{1}{\cos\theta + \sin\theta}} f(r\cos\theta, r\sin\theta) r dr + \int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^1 f(r\cos\theta, r\sin\theta) r dr.$$

故应选 D.



(4) A

$$\text{解 因为 } \int_{-\pi}^{\pi} (x - a\cos x - b\sin x)^2 dx = \frac{2}{3}\pi^3 + \pi(a^2 + b^2 - 4b).$$

所以相当于求 $a^2 + b^2 - 4b$ 极小值点.

显然 $a = 0, b = 2$ 时积分最小, 即 $a_1 \cos x + b_1 \sin x = 2 \sin x$. 故应选 A.

(5) B

解 由行列式展开定理按第一列展开:

$$\begin{vmatrix} 0 & a & b & 0 \\ a & 0 & 0 & b \\ 0 & c & d & 0 \\ c & 0 & 0 & d \end{vmatrix} = -a \begin{vmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & d \end{vmatrix} - c \begin{vmatrix} a & b & 0 \\ 0 & 0 & b \\ c & d & 0 \end{vmatrix} = -ad \begin{vmatrix} a & b \\ c & d \end{vmatrix} + bc \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ = -ad(ad - bc) + bc(ad - bc) = -(ad - bc)^2. \quad \text{故应选 B.}$$

(6) A

$$\text{解 因为 } (\alpha_1 + k\alpha_3, \alpha_2 + l\alpha_3) = (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ k & l \end{pmatrix} = (\alpha_1, \alpha_2, \alpha_3) A.$$

对任意的常数 k, l , 矩阵 A 的秩都为 2,

所以若向量 $\alpha_1, \alpha_2, \alpha_3$ 线性无关, 则 $\alpha_1 + k\alpha_3, \alpha_2 + l\alpha_3$ 一定线性无关.

而当 $\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ 时,

对任意的常数 k, l , 向量 $\alpha_1 + k\alpha_3, \alpha_2 + l\alpha_3$ 线性无关, 但 $\alpha_1, \alpha_2, \alpha_3$ 线性相关. 故应选 A.

(7) B

解 $P(A - B) = P(A) - P(AB) = P(A) - P(A)P(B) = P(A) - 0.5P(A)$
 $= 0.5P(A) = 0.3,$

得 $P(A) = 0.6,$

则 $P(B - A) = P(B) - P(AB) = P(B) - P(A)P(B) = 0.2.$ 故应选 B.

(8) D

解 $EY_1 = \int_{-\infty}^{+\infty} y f_{Y_1}(y) dy = \frac{1}{2} \left[\int_{-\infty}^{+\infty} y f_1(y) dy + \int_{-\infty}^{+\infty} y f_2(y) dy \right] = \frac{1}{2} (EX_1 + EX_2),$

$$EY_2 = \frac{1}{2} E(X_1 + X_2) = \frac{1}{2} (EX_1 + EX_2),$$

故 $EY_1 = EY_2$, 又因为

$$DY_1 = E(Y_1^2) - (EY_1)^2, DY_2 = E(Y_2^2) - (EY_2)^2,$$

则 $DY_1 - DY_2 = E(Y_1^2) - E(Y_2^2)$

$$= \frac{1}{2} \left[\int_{-\infty}^{+\infty} y^2 f_1(y) dy + \int_{-\infty}^{+\infty} y^2 f_2(y) dy \right] - E \left[\frac{1}{4} (X_1 + X_2)^2 \right]$$

$$= \frac{1}{2} E(X_1^2) + \frac{1}{2} E(X_2^2) - \frac{1}{4} E[(X_1 + X_2)^2]$$

$$= \frac{1}{4} E(X_1^2 + X_2^2 - 2X_1X_2) = \frac{1}{4} E[(X_1 - X_2)^2] > 0,$$

即 $DY_1 > DY_2$. 故应选 D.

二、填空题

(9) $2x - y - z = 1$

解 $Z'_x = 2x(1 - \sin y) - \cos x \cdot y^2, Z'_x(1, 0) = 2.$

$$Z'_y = -x^2 \cos y + 2y(1 - \sin x), Z'_y(1, 0) = -1.$$

所以曲面在 $(1, 0, 1)$ 处的法向量为 $\mathbf{n} = \{2, -1, -1\}.$

则切平面方程为 $2(x - 1) + (-1)(y - 0) + (-1)(z - 1) = 0,$

即 $2x - y - z = 1.$

(10) 1

解 $f(x) = \int 2(x - 1) dx = x^2 - 2x + C, x \in [0, 2].$

因为 $f(x)$ 为奇函数, 所以 $f(0) = 0$, 可知 $C = 0,$

即 $f(x) = x^2 - 2x.$

又 $f(x)$ 的周期为 4, 故 $f(7) = f(-1 + 8) = f(-1) = -f(1) = 1.$

(11) $x e^{2x+1}$

解 方程变形为 $y' = \frac{y}{x} \ln \frac{y}{x}$, 属于齐次方程.

设 $u = \frac{y}{x}$, 则 $u + x \frac{du}{dx} = u \ln u,$

分离变量得

$$\frac{du}{u(\ln u - 1)} = \frac{1}{x} dx,$$

两边积分得

$$\int \frac{du}{u(\ln u - 1)} = \int \frac{1}{x} dx,$$

$$\ln |\ln u - 1| = \ln x + C_1,$$

即 $\ln u - 1 = Cx$.

故 $\ln \frac{y}{x} - 1 = Cx$, 代入 $y(1) = e^3$, 可得 $C = 2$, 所以 $y = x e^{2x+1}$.

(12) π

解 由斯托克斯公式

$$\oint_L z dx + y dz = \iint_{\Sigma} \begin{vmatrix} dy dz & dz dx & dx dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 0 & y \end{vmatrix} = \iint_{\Sigma} dy dz + dz dx = \iint_{\Sigma} dx dy = \iint_{D_{xy}} dx dy = \pi,$$

其中 $\Sigma: \begin{cases} x^2 + y^2 \leq 1, \\ y + z = 0 \end{cases}$, 取上侧, $D_{xy} = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

(13) $[-2, 2]$

解 由配方法可知 $f(x_1, x_2, x_3) = x_1^2 - x_2^2 + 2ax_1x_3 + 4x_2x_3$
 $= (x_1 + ax_3)^2 - (x_2 - 2x_3)^2 + (4 - a^2)x_3^2$.

由于负惯性指数为 1, 则 $4 - a^2 \geq 0$,

所以 a 的取值范围是 $[-2, 2]$.

(14) $\frac{2}{5n}$

$$\begin{aligned} \text{解 } E\left(C \sum_{i=1}^n X_i^2\right) &= C \sum_{i=1}^n E(X_i^2) = C \sum_{i=1}^n E(X^2) \\ &= Cn \int_{-\infty}^{+\infty} x^2 f(x) dx = Cn \int_{\theta}^{2\theta} x^2 \cdot \frac{2x}{3\theta^2} dx = Cn \cdot \frac{5}{2} \theta^2. \end{aligned}$$

因为 $C \sum_{i=1}^n X_i^2$ 是 θ^2 的无偏性估计, 所以 $E\left(C \sum_{i=1}^n X_i^2\right) = \theta^2$.

即 $Cn \cdot \frac{5}{2} \theta^2 = \theta^2$, 所以 $C = \frac{2}{5n}$.

三、解答题

$$\begin{aligned} (15) \text{ 解 } \lim_{x \rightarrow +\infty} \frac{\int_1^x [t^2(e^{\frac{1}{t}} - 1) - t] dt}{x^2 \ln\left(1 + \frac{1}{x}\right)} &= \lim_{x \rightarrow +\infty} \frac{\int_1^x [t^2(e^{\frac{1}{t}} - 1) - t] dt}{x} \\ &= \lim_{x \rightarrow +\infty} [x^2(e^{\frac{1}{x}} - 1) - x] \\ &= \lim_{x \rightarrow +\infty} \left[x^2 \left(\frac{1}{x} + \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right) \right) - x \right] \\ &= \lim_{x \rightarrow +\infty} \left[\frac{1}{2} + x^2 \cdot o\left(\frac{1}{x^2}\right) \right] \\ &= \frac{1}{2}. \end{aligned}$$

(16) 解 在 $y^3 + xy^2 + x^2y + 6 = 0$ 两端关于 x 求导, 得

$$3y^2y' + y^2 + 2xyy' + 2xy + x^2y' = 0.$$

令 $y' = 0$, 得 $y = -2x$, 或 $y = 0$ (不适合方程, 舍去).

将 $y = -2x$ 代入方程得 $-6x^3 + 6 = 0$, 解得 $x = 1$, $f(1) = -2$.

在 $3y^2y' + y^2 + 2xyy' + 2xy + x^2y' = 0$ 两端关于 x 求导, 得
($3y^2 + 2xy + x^2$) $y'' + 2(3y + x)(y')^2 + 4(y + x)y' + 2y = 0$.

求得 $f''(1) = \frac{4}{9} > 0$.

所以 $x = 1$ 是函数 $f(x)$ 的极小值点, 极小值为 $f(1) = -2$.

(17) 解 因为 $\frac{\partial z}{\partial x} = f'(e^x \cos y)e^x \cos y$,

$$\frac{\partial^2 z}{\partial x^2} = f''(e^x \cos y)e^{2x} \cos^2 y + f'(e^x \cos y)e^x \cos y,$$

$$\frac{\partial z}{\partial y} = -f'(e^x \cos y)e^x \sin y,$$

$$\frac{\partial^2 z}{\partial y^2} = f''(e^x \cos y)e^{2x} \sin^2 y - f'(e^x \cos y)e^x \cos y,$$

所以 $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (4z + e^x \cos y)e^{2x}$ 化为

$$f''(e^x \cos y)e^{2x} = [4f(e^x \cos y) + e^x \cos y]e^{2x}.$$

从而函数 $f(u)$ 满足方程

$$f''(u) = 4f(u) + u. \quad \textcircled{1}$$

方程 ① 对应的齐次方程的通解为

$$f(u) = C_1 e^{2u} + C_2 e^{-2u}.$$

方程 ① 的一个特解为 $-\frac{u}{4}$, 故方程 ① 的通解为

$$f(u) = C_1 e^{2u} + C_2 e^{-2u} - \frac{u}{4}.$$

由 $f(0) = 0, f'(0) = 0$ 得 $\begin{cases} C_1 + C_2 = 0, \\ 2C_1 - 2C_2 - \frac{1}{4} = 0. \end{cases}$

解得 $C_1 = \frac{1}{16}, C_2 = -\frac{1}{16}$.

故 $f(u) = \frac{1}{16}(e^{2u} - e^{-2u} - 4u)$.

(18) 解 设 Σ_1 为平面 $z = 1$ 上被 $\begin{cases} x^2 + y^2 = 1, \\ z = 1 \end{cases}$ 所围部分的下侧, Σ_1 与 Σ 所围成的空间区域记为 Ω , 则

$$\begin{aligned} & \oiint_{\Sigma + \Sigma_1} (x-1)^3 dy dz + (y-1)^3 dz dx + (z-1) dx dy \\ &= - \iiint_{\Omega} [3(x-1)^2 + 3(y-1)^2 + 1] dx dy dz \end{aligned}$$

由于 $\iiint_{\Sigma_1} (x-1)^3 dy dz + (y-1)^3 dz dx + (z-1) dx dy = 0$

$$\iiint_{\Omega} x \, dx \, dy \, dz = \iiint_{\Omega} y \, dx \, dy \, dz = 0,$$

所以
$$I = - \iiint_{\Omega} (3x^2 + 3y^2 + 7) \, dx \, dy \, dz.$$

$$\iiint_{\Omega} (3x^2 + 3y^2 + 7) \, dx \, dy \, dz = \int_0^{2\pi} d\theta \int_0^1 dr \int_{r^2}^1 (3r^2 + 7)r \, dz = 2\pi \int_0^1 r(1-r^2)(3r^2 + 7) \, dr = 4\pi,$$

于是 $I = -4\pi$.

(19) 证 (I) 因为 $\cos a_n - \cos b_n = a_n$, 且 $0 < a_n < \frac{\pi}{2}, 0 < b_n < \frac{\pi}{2}$, 所以 $0 < a_n < b_n$.

又因为 $\sum_{n=1}^{\infty} b_n$ 收敛, 所以 $\lim_{n \rightarrow \infty} b_n = 0$.

故 $\lim_{n \rightarrow \infty} a_n = 0$.

(II) 因为
$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n^2} &= \lim_{n \rightarrow \infty} \frac{1 - \cos b_n}{b_n^2} \cdot \frac{a_n}{1 - \cos b_n} \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \frac{a_n}{1 - \cos b_n} \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \frac{a_n}{a_n + 1 - \cos a_n} \\ &= \frac{1}{2}, \end{aligned}$$

且级数 $\sum_{n=1}^{\infty} b_n$ 收敛, 所以 $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ 收敛.

(20) 解 (I) 对矩阵 A 施以初等行变换

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & 0 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{pmatrix},$$

则方程组 $Ax = 0$ 的一个基础解系为 $\alpha = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$.

(II) 对矩阵 $(A : E)$ 施以初等行变换

$$(A : E) = \left(\begin{array}{cccc|cccc} 1 & -2 & 3 & -4 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & -3 & 0 & 0 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 2 & 6 & -1 \\ 0 & 1 & 0 & -2 & -1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -3 & -1 & -4 & 1 & 0 \end{array} \right).$$

记 $E = (e_1, e_2, e_3)$, 则

$$Ax = e_1 \text{ 的通解为 } x = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \end{pmatrix} + k_1 \alpha, \quad k_1 \text{ 为任意常数};$$

$$Ax = e_2 \text{ 的通解为 } x = \begin{pmatrix} 6 \\ -3 \\ -4 \\ 0 \end{pmatrix} + k_2 \alpha, \quad k_2 \text{ 为任意常数};$$

$$\mathbf{A}\mathbf{x} = \mathbf{e}_3 \text{ 的通解为 } \mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + k_3 \boldsymbol{\alpha}, \quad k_3 \text{ 为任意常数.}$$

于是, 所求矩阵为

$$\mathbf{B} = \begin{pmatrix} 2 & 6 & -1 \\ -1 & -3 & 1 \\ -1 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix} + (k_1 \boldsymbol{\alpha}, k_2 \boldsymbol{\alpha}, k_3 \boldsymbol{\alpha}), \quad k_1, k_2, k_3 \text{ 为任意常数.}$$

$$(21) \text{ 证 } \text{ 设 } \mathbf{A} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 2 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & n \end{pmatrix}.$$

因为

$$|\lambda \mathbf{E} - \mathbf{A}| = \begin{vmatrix} \lambda - 1 & -1 & \cdots & -1 \\ -1 & \lambda - 1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & \lambda - 1 \end{vmatrix} = (\lambda - n)\lambda^{n-1},$$

$$|\lambda \mathbf{E} - \mathbf{B}| = \begin{vmatrix} \lambda & 0 & \cdots & -1 \\ 0 & \lambda & \cdots & -2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda - n \end{vmatrix} = (\lambda - n)\lambda^{n-1},$$

所以 $\mathbf{A}$ 与 $\mathbf{B}$ 有相同的特征值 $\lambda_1 = n, \lambda_2 = 0 (n-1 \text{ 重})$.

由于 $\mathbf{A}$ 为实对称矩阵, 所以 $\mathbf{A}$ 相似于对角矩阵

$$\boldsymbol{\Lambda} = \begin{pmatrix} n & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

因为 $r(\lambda_2 \mathbf{E} - \mathbf{B}) = r(\mathbf{B}) = 1$,

所以 $\mathbf{B}$ 对应于特征值 $\lambda_2 = 0$ 有 $n-1$ 个线性无关的特征向量,

于是 $\mathbf{B}$ 也相似于 $\boldsymbol{\Lambda}$.

故 $\mathbf{A}$ 与 $\mathbf{B}$ 相似.

$$\begin{aligned} (22) \text{ 解 } \quad (\text{I}) \quad F_Y(y) &= P\{Y \leq y\} \\ &= P\{X=1\}P\{Y \leq y \mid X=1\} + P\{X=2\}P\{Y \leq y \mid X=2\} \\ &= \frac{1}{2}P\{Y \leq y \mid X=1\} + \frac{1}{2}P\{Y \leq y \mid X=2\}. \end{aligned}$$

当 $y < 0$ 时, $F_Y(y) = 0$;

当 $0 \leq y < 1$ 时, $F_Y(y) = \frac{3y}{4}$;

当 $1 \leq y < 2$ 时, $F_Y(y) = \frac{1}{2} + \frac{y}{4}$;

当 $y \geq 2$ 时, $F_Y(y) = 1$.

所以 Y 的分布函数为

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ \frac{3y}{4}, & 0 \leq y < 1, \\ \frac{1}{2} + \frac{y}{4}, & 1 \leq y < 2, \\ 1, & y \geq 2. \end{cases}$$

(II) 随机变量 Y 的概率密度为

$$f_Y(y) = \begin{cases} \frac{3}{4}, & 0 < y < 1, \\ \frac{1}{4}, & 1 \leq y < 2, \\ 0, & \text{其他.} \end{cases}$$

$$EY = \int_{-\infty}^{+\infty} y f_Y(y) dy = \int_0^1 \frac{3}{4} y dy + \int_1^2 \frac{1}{4} y dy = \frac{3}{4}.$$

(23) 解 (I) 总体 X 的概率密度为 $f(x; \theta) = \begin{cases} \frac{2x}{\theta} e^{-\frac{x^2}{\theta}}, & x \geq 0, \\ 0, & x < 0. \end{cases}$

$$EX = \int_0^{+\infty} x \cdot \frac{2x}{\theta} e^{-\frac{x^2}{\theta}} dx = - \int_0^{+\infty} x d e^{-\frac{x^2}{\theta}} = \int_0^{+\infty} e^{-\frac{x^2}{\theta}} dx = \frac{\sqrt{\pi\theta}}{2} \cdot \frac{1}{\sqrt{\pi\theta}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{\theta}} dx = \frac{\sqrt{\pi\theta}}{2},$$

$$EX^2 = \int_0^{+\infty} x^2 \cdot \frac{2x}{\theta} e^{-\frac{x^2}{\theta}} dx = \theta \int_0^{+\infty} u e^{-u} du = \theta.$$

(II) 设 $x_1, x_2, \dots, x_n$ 为样本观测值, 似然函数为

$$L(\theta) = \prod_{i=1}^n f(x_i) = \begin{cases} \frac{2^n x_1 x_2 \cdots x_n}{\theta^n} e^{-\frac{1}{\theta} \sum_{i=1}^n x_i^2}, & x_1, x_2, \dots, x_n > 0, \\ 0, & \text{其他.} \end{cases}$$

当 $x_1, x_2, \dots, x_n > 0$ 时, $\ln L(\theta) = n \ln 2 + \sum_{i=1}^n \ln x_i - n \ln \theta - \frac{1}{\theta} \sum_{i=1}^n x_i^2$.

令 $\frac{d \ln L(\theta)}{d \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n x_i^2 = 0$, 得 θ 的最大似然估计值为 $\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n x_i^2$.

从而 θ 的最大似然估计量为

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n X_i^2.$$

(III) 存在, $a = \theta$. 因为 $\{X_n^2\}$ 是独立同分布的随机变量序列, 且 $EX_1^2 = \theta < +\infty$, 所以根据

辛钦大数定律, 当 $n \rightarrow \infty$ 时, $\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n X_i^2$ 依概率收敛于 EX_1^2 , 即 θ . 所以对任何 $\varepsilon > 0$ 都有

$$\lim_{n \rightarrow \infty} P\{|\hat{\theta}_n - \theta| \geq \varepsilon\} = 0.$$