

# 2014 年

## 数学(三) 参考答案

### 一、选择题

(1) A

解 因为  $\lim_{n \rightarrow \infty} a_n = a \neq 0$ , 所以  $\forall \varepsilon > 0, \exists N$ , 当  $n > N$  时,

有  $|a_n - a| < \varepsilon$ , 即  $a - \varepsilon < a_n < a + \varepsilon$ , 则  $|a| - \varepsilon < |a_n| \leq |a| + \varepsilon$ ,

取  $\varepsilon = \frac{|a|}{2}$ , 则知  $|a_n| > \frac{|a|}{2}$ . 故应选 A.

(2) C

解 由渐近线定义可知, 四个选项的曲线均不存在水平渐近线和垂直渐近线.

对于  $y = x + \sin \frac{1}{x}$ , 可知  $a = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{x + \sin \frac{1}{x}}{x} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \sin \frac{1}{x}\right) = 1$ ,

$$b = \lim_{x \rightarrow \infty} [f(x) - ax] = \lim_{x \rightarrow \infty} \sin \frac{1}{x} = 0.$$

所以  $y = x$  是  $y = x + \sin \frac{1}{x}$  的斜渐近线. 故应选 C.

(3) D

解 因为  $p(x) - \tan x = a + bx + cx^2 + dx^3 - \tan x$  是无穷小 ( $x \rightarrow 0$  时), 所以  $a = 0$ .

$$\text{又 } \lim_{x \rightarrow 0} \frac{p(x) - \tan x}{x^3} = \lim_{x \rightarrow 0} \frac{bx + cx^2 + dx^3 - \tan x}{x^3} = \lim_{x \rightarrow 0} \left( \frac{b + 2cx - \sec^2 x}{3x^2} + d \right).$$

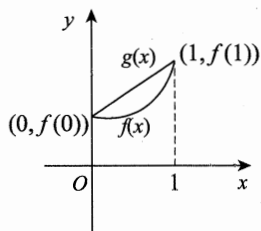
所以  $b = 1$ , 原式  $= \lim_{x \rightarrow 0} \left( \frac{2c}{3x} - \frac{1}{3} + d \right) = 0$ , 则  $c = 0, d = \frac{1}{3}$ . 故应选 D.

(4) D

解 当  $f''(x) \geq 0$  时,  $f(x)$  是凹函数.

而  $g(x) = [f(1) - f(0)]x + f(0)$  可视为连接  $(0, f(0))$  与  $(1, f(1))$  的直线段, 如右图所示, 则  $f(x) \leq g(x)$ .

故应选 D.



(5) B

解 由行列式展开定理按第一列展开:

$$\begin{vmatrix} 0 & a & b & 0 \\ a & 0 & 0 & b \\ 0 & c & d & 0 \\ c & 0 & 0 & d \end{vmatrix} = -a \begin{vmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & d \end{vmatrix} - c \begin{vmatrix} a & b & 0 \\ 0 & 0 & b \\ c & d & 0 \end{vmatrix} = -ad \begin{vmatrix} a & b \\ c & d \end{vmatrix} + bc \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ = -ad(ad - bc) + bc(ad - bc) = -(ad - bc)^2. \quad \text{故应选 B.}$$

(6) A

解 因为  $(\alpha_1 + k\alpha_3, \alpha_2 + l\alpha_3) = (\alpha_1, \alpha_2, \alpha_3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ k & l \end{bmatrix} = (\alpha_1, \alpha_2, \alpha_3)A$ .

对任意的常数  $k, l$ , 矩阵  $A$  的秩都为 2,

所以若向量  $\alpha_1, \alpha_2, \alpha_3$  线性无关, 则  $\alpha_1 + k\alpha_3, \alpha_2 + l\alpha_3$  一定线性无关.

而当  $\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  时,

对任意的常数  $k, l$ , 向量  $\alpha_1 + k\alpha_3, \alpha_2 + l\alpha_3$  线性无关, 但  $\alpha_1, \alpha_2, \alpha_3$  线性相关. 故应选 A.

(7) B

解  $P(A - B) = P(A) - P(AB) = P(A) - P(A)P(B) = P(A) - 0.5P(A)$   
 $= 0.5P(A) \stackrel{\text{令}}{=} 0.3,$

得  $P(A) = 0.6,$

则  $P(B - A) = P(B) - P(AB) = P(B) - P(A)P(B) = 0.2.$  故应选 B.

(8) C

解 因为  $X_i \sim N(0, \sigma^2), i = 1, 2, 3$ , 所以  $X_1 - X_2 \sim N(0, 2\sigma^2)$ , 则

$$\frac{X_1 - X_2}{\sqrt{2}\sigma} \sim N(0, 1),$$

又  $\frac{X_3}{\sigma} \sim N(0, 1)$ , 则  $\left(\frac{X_3}{\sigma}\right)^2 \sim \chi^2(1)$ . 由  $t$  分布定义, 知

$$\frac{\frac{X_1 - X_2}{\sqrt{2}\sigma}}{\sqrt{\left(\frac{X_3}{\sigma}\right)^2}} = \frac{X_1 - X_2}{\sqrt{2}|X_3|} \sim t(1). \quad \text{故应选 C.}$$

## 二、填空题

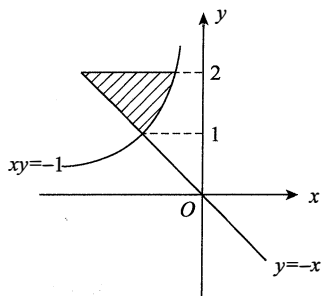
(9)  $20 - Q$

解 收益函数  $R(Q) = PQ = 20Q - \frac{1}{2}Q^2$ , 所以边际收益为  $R'(Q) = 20 - Q$ .

(10)  $\frac{3}{2} - \ln 2$

解 区域  $D$  的图形如右图所示, 面积

$$\begin{aligned} S &= \int_1^2 \left[ -\frac{1}{y} - (-y) \right] dy \\ &= \left( \frac{y^2}{2} - \ln y \right) \Big|_1^2 \\ &= \frac{3}{2} - \ln 2. \end{aligned}$$



(11)  $\frac{1}{2}$

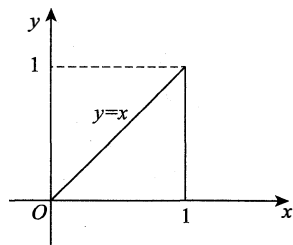
解 由于  $\int_0^a x e^{2x} dx = \frac{e^{2x}}{4} (2x - 1) \Big|_0^a = \frac{e^{2a}}{4} (2a - 1) + \frac{1}{4} \stackrel{\text{令}}{=} \frac{1}{4};$

得  $a = \frac{1}{2}.$

(12)  $\frac{e-1}{2}$

解 如右图所示,则

$$\begin{aligned} & \int_0^1 dy \int_y^1 \left( \frac{e^{x^2}}{x} - e^{y^2} \right) dx \\ &= \int_0^1 dy \int_y^1 \frac{e^{x^2}}{x} dx - \int_0^1 dy \int_y^1 e^{y^2} dx \\ &= \int_0^1 dx \int_0^x \frac{e^{x^2}}{x} dy - \int_0^1 e^{y^2} (1-y) dy \\ &= \int_0^1 e^{x^2} dx - \int_0^1 e^{y^2} dy + \int_0^1 y e^{y^2} dy = \frac{1}{2} e^{y^2} \Big|_0^1 = \frac{1}{2} (e - 1). \end{aligned}$$



(13)  $[-2, 2]$

解 由配方法可知

$$\begin{aligned} f(x_1, x_2, x_3) &= x_1^2 - x_2^2 + 2ax_1x_3 + 4x_2x_3 \\ &= (x_1 + ax_3)^2 - (x_2 - 2x_3)^2 + (4 - a^2)x_3^2. \end{aligned}$$

由于负惯性指数为 1, 则  $4 - a^2 \geq 0$ , 所以  $a$  的取值范围是  $[-2, 2]$ .

(14)  $\frac{2}{5n}$

解  $E(x^2) = \int_{-\infty}^{+\infty} x^2 f(x) dx = \int_{\theta}^{2\theta} x^2 \cdot \frac{2x}{3\theta^2} dx = \frac{2}{3\theta^2} \int_{\theta}^{2\theta} x^3 dx = \frac{5}{2} \theta^2$ , 则

$$E\left(c \sum_{i=1}^n X_i^2\right) = c \sum_{i=1}^n E(X_i^2) = c \sum_{i=1}^n E(X^2) = cn \cdot \frac{5}{2} \theta^2 \stackrel{\text{令}}{=} \theta^2.$$

所以  $c = \frac{2}{5n}$ .

### 三、解答题

(15) 解  $\lim_{x \rightarrow +\infty} \frac{\int_1^x [t^2(e^{\frac{1}{t}} - 1) - t] dt}{x^2 \ln\left(1 + \frac{1}{x}\right)} = \lim_{x \rightarrow +\infty} \frac{\int_1^x [t^2(e^{\frac{1}{t}} - 1) - t] dt}{x}$

$$= \lim_{x \rightarrow +\infty} [x^2(e^{\frac{1}{x}} - 1) - x]$$

$$\stackrel{\text{令 } u = \frac{1}{x}}{=} \lim_{u \rightarrow 0^+} \frac{e^u - 1 - u}{u^2}$$

$$= \lim_{u \rightarrow 0^+} \frac{e^u - 1}{2u} = \frac{1}{2}.$$

(16) 解  $\iint_D \frac{x \sin(\pi \sqrt{x^2 + y^2})}{x + y} dx dy = \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\cos \theta + \sin \theta} d\theta \cdot \int_1^2 r \sin \pi r dr.$

$$\begin{aligned} \text{由于 } \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\cos \theta + \sin \theta} d\theta &= \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\cos \theta + \sin \theta} d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos \theta + \sin \theta}{\cos \theta + \sin \theta} d\theta \\ &= \frac{\pi}{4}, \end{aligned}$$

$$\int_1^2 r \sin \pi r dr = \frac{1}{\pi} \left( -r \cos \pi r + \frac{1}{\pi} \sin \pi r \right) \Big|_1^2 = -\frac{3}{\pi},$$

$$\text{故} \iint_D \frac{x \sin(\pi \sqrt{x^2 + y^2})}{x + y} dx dy = -\frac{3}{4}.$$

(17) 解 因为  $\frac{\partial z}{\partial x} = f'(e^x \cos y) e^x \cos y$ ,  $\frac{\partial z}{\partial y} = -f'(e^x \cos y) e^x \sin y$ ,

所以  $\cos y \frac{\partial z}{\partial x} - \sin y \frac{\partial z}{\partial y} = (4z + e^x \cos y) e^x$  化为

$$f'(e^x \cos y) e^x = [4f(e^x \cos y) + e^x \cos y] e^x.$$

即函数  $f(u)$  满足方程

$$f'(u) - 4f(u) = u,$$

该方程的通解为

$$f(u) = C e^{4u} - \frac{u}{4} - \frac{1}{16}.$$

又  $f(0) = 0$ , 得  $C = \frac{1}{16}$ ,

故  $f(u) = \frac{1}{16}(e^{4u} - 4u - 1)$ .

(18) 解 幂级数的系数  $a_n = (n+1)(n+3)$ .

因为  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{(n+2)(n+4)}{(n+1)(n+3)} = 1$ , 所以收敛半径  $R = 1$ .

当  $x = \pm 1$  时, 因级数  $\sum_{n=0}^{\infty} (n+1)(n+3)$  及  $\sum_{n=0}^{\infty} (n+1)(n+3)(-1)^n$  发散, 故收敛域为  $(-1, 1)$ .

设  $S(x) = \sum_{n=0}^{\infty} (n+1)(n+3)x^n$ ,  $x \in (-1, 1)$ ,

$$\text{则} \int_0^x S(t) dt = \sum_{n=0}^{\infty} (n+3)x^{n+1} = \sum_{n=0}^{\infty} (n+2)x^{n+1} + \sum_{n=0}^{\infty} x^{n+1},$$

$$\text{其中} \sum_{n=0}^{\infty} x^{n+1} = \frac{x}{1-x},$$

$$\sum_{n=0}^{\infty} (n+2)x^{n+1} = \left( \sum_{n=0}^{\infty} \int_0^x (n+2)t^{n+1} dt \right)' = \left( \frac{x^2}{1-x} \right)' = \frac{2x - x^2}{(1-x)^2},$$

$$\text{所以} S(x) = \left( \frac{2x - x^2}{(1-x)^2} \right)' + \left( \frac{x}{1-x} \right)' = \frac{3-x}{(1-x)^3}, x \in (-1, 1).$$

(19) 证 (I) 因为  $0 \leq g(x) \leq 1$ ,

所以当  $x \in [a, b]$  时, 有  $\int_a^x 0 dt \leq \int_a^x g(t) dt \leq \int_a^x 1 dt$ ,

$$\text{即} 0 \leq \int_a^x g(t) dt \leq x - a.$$

$$(II) \text{ 令 } F(x) = \int_a^{a+\int_a^x g(u) du} f(t) dt - \int_a^x f(t) g(t) dt, x \in [a, b].$$

因为  $f(x), g(x)$  在区间  $[a, b]$  上连续,

所以  $F(x)$  在区间  $[a, b]$  上可导, 且

$$F'(x) = \left[ f\left(a + \int_a^x g(u) du\right) - f(x) \right] g(x).$$

由(I)知,  $a + \int_a^x g(u)du \leq x$ ,

又因为  $f(x)$  单调增加, 且  $g(x) \geq 0$ ,

所以  $F'(x) \leq 0$ , 从而  $F(x)$  在区间  $[a, b]$  上单调减少.

又  $F(a) = 0$ , 故  $F(b) \leq 0$ , 即

$$\int_a^{a+\int_a^b g(x)dx} f(x)dx \leq \int_a^b f(x)g(x)dx.$$

(20) 解 (I) 对矩阵  $A$  施以初等行变换

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & 0 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{pmatrix},$$

则方程组  $Ax = 0$  的一个基础解系为  $\alpha = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$ .

(II) 对矩阵  $(A : E)$  施以初等行变换

$$(A : E) = \left( \begin{array}{cccc|cccc} 1 & -2 & 3 & -4 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & -3 & 0 & 0 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 2 & 6 & -1 & 0 \\ 0 & 1 & 0 & -2 & -1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -3 & -1 & -4 & 1 & 0 \end{array} \right).$$

记  $E = (e_1, e_2, e_3)$ , 则

$$Ax = e_1 \text{ 的通解为 } x = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \end{pmatrix} + k_1 \alpha, \quad k_1 \text{ 为任意常数};$$

$$Ax = e_2 \text{ 的通解为 } x = \begin{pmatrix} 6 \\ -3 \\ -4 \\ 0 \end{pmatrix} + k_2 \alpha, \quad k_2 \text{ 为任意常数};$$

$$Ax = e_3 \text{ 的通解为 } x = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + k_3 \alpha, \quad k_3 \text{ 为任意常数}.$$

于是, 所求矩阵为

$$B = \begin{pmatrix} 2 & 6 & -1 \\ -1 & -3 & 1 \\ -1 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix} + (k_1 \alpha, k_2 \alpha, k_3 \alpha), \quad k_1, k_2, k_3 \text{ 为任意常数}.$$

$$(21) \text{ 证 设 } A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 2 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & n \end{pmatrix}.$$

因为

$$|\lambda \mathbf{E} - \mathbf{A}| = \begin{vmatrix} \lambda - 1 & -1 & \cdots & -1 \\ -1 & \lambda - 1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & \lambda - 1 \end{vmatrix} = (\lambda - n)\lambda^{n-1},$$

$$|\lambda \mathbf{E} - \mathbf{B}| = \begin{vmatrix} \lambda & 0 & \cdots & -1 \\ 0 & \lambda & \cdots & -2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda - n \end{vmatrix} = (\lambda - n)\lambda^{n-1},$$

所以  $\mathbf{A}$  与  $\mathbf{B}$  有相同的特征值  $\lambda_1 = n, \lambda_2 = 0 (n-1 \text{ 重})$ .

由于  $\mathbf{A}$  为实对称矩阵, 所以  $\mathbf{A}$  相似于对角矩阵

$$\mathbf{A} = \begin{pmatrix} n & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

因为  $r(\lambda_2 \mathbf{E} - \mathbf{B}) = r(\mathbf{B}) = 1$ ,

所以  $\mathbf{B}$  对应于特征值  $\lambda_2 = 0$  有  $n-1$  个线性无关的特征向量,

于是  $\mathbf{B}$  也相似于  $\mathbf{A}$ .

故  $\mathbf{A}$  与  $\mathbf{B}$  相似.

$$\begin{aligned} (22) \text{ 解 } (I) F_Y(y) &= P\{Y \leq y\} \\ &= P\{X=1\}P\{Y \leq y \mid X=1\} + P\{X=2\}P\{Y \leq y \mid X=2\} \\ &= \frac{1}{2}P\{Y \leq y \mid X=1\} + \frac{1}{2}P\{Y \leq y \mid X=2\}. \end{aligned}$$

当  $y < 0$  时,  $F_Y(y) = 0$ ;

当  $0 \leq y < 1$  时,  $F_Y(y) = \frac{3y}{4}$ ;

当  $1 \leq y < 2$  时,  $F_Y(y) = \frac{1}{2} + \frac{y}{4}$ ;

当  $y \geq 2$  时,  $F_Y(y) = 1$ .

所以  $Y$  的分布函数为

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ \frac{3y}{4}, & 0 \leq y < 1, \\ \frac{1}{2} + \frac{y}{4}, & 1 \leq y < 2, \\ 1, & y \geq 2. \end{cases}$$

(II) 随机变量  $Y$  的概率密度为

$$f_Y(y) = \begin{cases} \frac{3}{4}, & 0 < y < 1, \\ \frac{1}{4}, & 1 \leq y < 2, \\ 0, & \text{其他.} \end{cases}$$

$$EY = \int_{-\infty}^{+\infty} y f_Y(y) dy = \int_0^1 \frac{3}{4} y dy + \int_1^2 \frac{1}{4} y dy = \frac{3}{4}.$$

(23) 解 (I) 设  $(X, Y)$  的概率分布为

| X \ Y | 0   | 1   |
|-------|-----|-----|
| 0     | $a$ | $b$ |
| 1     | $c$ | $d$ |

由题设条件知

$$EX = EY = \frac{2}{3}, DX = DY = \frac{2}{3} \left(1 - \frac{2}{3}\right) = \frac{2}{9}.$$

$$\text{Cov}(X, Y) = E(XY) - EX \cdot EY = P\{X=1, Y=1\} - \frac{4}{9}.$$

由

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sqrt{DX \cdot DY}} = \frac{P\{X=1, Y=1\} - \frac{4}{9}}{\frac{2}{9}} = \frac{1}{2}.$$

解得

$$d = P\{X=1, Y=1\} = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}.$$

由此可得

$$c = b = \frac{2}{3} - d = \frac{1}{9}, a = \frac{1}{3} - b = \frac{2}{9}.$$

所以  $(X, Y)$  的概率分布为

| X \ Y | 0             | 1             |
|-------|---------------|---------------|
| 0     | $\frac{2}{9}$ | $\frac{1}{9}$ |
| 1     | $\frac{1}{9}$ | $\frac{5}{9}$ |

$$(II) P\{X + Y \leq 1\} = 1 - P\{X + Y > 1\} = 1 - P\{X=1, Y=1\} = \frac{4}{9}.$$