

2011年(数一) 真题答案解析

一、选择题

(1) C

解 令 $u = (x-1)(x-2)^2, v = (x-3)^3(x-4)^4$,
 则 $y = uv, y' = u'v + uv', y'' = u''v + 2u'v' + uv'', y''' = u'''v + 3u''v' + 3u'v'' + uv'''$,
 由于已知函数 $y(x)$ 具有任意阶导数, 因此在拐点处必有 $y''(x) = 0$, 而 $y''(1) \neq 0, y''(2) \neq 0$,
 $y''(3) = 0, y''(4) = 0$, 于是可排除 A、B, 即点 $(3, 0)$ 与 $(4, 0)$ 是拐点的可疑点.
 而 $y'''(3) = 36 > 0$, 即 $y''(x)$ 在 $x = 3$ 的小邻域内单调增加, 由 $y''(3) = 0$ 知 $y''(x)$ 在 $x = 3$
 左、右两侧符号由负变为正, 即曲线 $y(x)$ 在点 $(3, 0)$ 两侧凹凸性相反, 由定义知点 $(3, 0)$ 为
 拐点; 而 $y''(x)$ 在 $x = 4$ 两侧符号相同, 则 $(4, 0)$ 不是拐点. 故应选 C.

(2) C

解 已知幂级数 $\sum_{n=1}^{\infty} a_n(x-1)^n$ 的收敛域必关于 $x = 1$ 对称(端点除外), 据此可排除选项 A、B;
 将 $x = 0$ 代入得级数 $\sum_{n=1}^{\infty} (-1)^n a_n$, 因为 $\{a_n\}$ 单调减少且 $\lim_{n \rightarrow \infty} a_n = 0$, 由莱布尼茨审敛法知级数
 $\sum_{n=1}^{\infty} (-1)^n a_n$ 收敛.
 将 $x = 2$ 代入得级数 $\sum_{n=1}^{\infty} a_n$, 其前 n 项和数列为: $S_n = \sum_{k=1}^n a_k$, 由条件知极限 $\lim_{n \rightarrow \infty} S_n$ 不存在, 故
 级数 $\sum_{n=1}^{\infty} a_n$ 发散,
 综上知: 幂级数 $\sum_{n=1}^{\infty} a_n(x-1)^n$ 的收敛域为 $[0, 2)$.

(3) A

解 $\frac{\partial z}{\partial x} = f'(x) \ln f(y), \frac{\partial z}{\partial y} = f(x) \cdot \frac{f'(y)}{f(y)},$

$\frac{\partial^2 z}{\partial x^2} = f''(x) \ln f(y), \frac{\partial^2 z}{\partial x \partial y} = f'(x) \cdot \frac{f'(y)}{f(y)},$

$\frac{\partial^2 z}{\partial y^2} = f(x) \frac{f''(y)f(y) - [f'(y)]^2}{f^2(y)},$

若函数 $z = f(x) \ln f(y)$ 在 $(0, 0)$ 处取得极小值, 则

$$\begin{cases} \left. \frac{\partial z}{\partial x} \right|_{(0,0)} = f'(0) \ln f(0) = 0, \\ \left. \frac{\partial z}{\partial y} \right|_{(0,0)} = f(0) \cdot \frac{f'(0)}{f(0)} = 0, \end{cases} \quad ①$$

$$\begin{aligned} \text{且 } B^2 - AC &= \left(\left. \frac{\partial^2 z}{\partial x \partial y} \right|_{(0,0)} \right)^2 - \left. \frac{\partial^2 z}{\partial x^2} \right|_{(0,0)} \cdot \left. \frac{\partial^2 z}{\partial y^2} \right|_{(0,0)} \\ &= \frac{[f'(0)]^4}{f^2(0)} - f''(0) \ln f(0) \cdot f(0) \frac{f(0)f''(0) - [f'(0)]^2}{f^2(0)} < 0, \end{aligned} \quad ②$$

$$A = \left. \frac{\partial^2 z}{\partial x^2} \right|_{(0,0)} = f''(0) \ln f(0) > 0, \quad ③$$

由①、②、③式得 $f(0) > 1, f''(0) > 0$, 故应选 A.

(4) B

解 当 $0 < x < \frac{\pi}{4}$ 时, $\sin x < \cos x < 1 < \cot x$, 于是 $\ln \sin x < \ln \cos x < \ln \cot x$,

由定积分性质得

$$\int_0^{\frac{\pi}{4}} \ln(\sin x) dx < \int_0^{\frac{\pi}{4}} \ln(\cos x) dx < \int_0^{\frac{\pi}{4}} \ln(\cot x) dx,$$

即 $I < K < J$. 故应选 B.

(5) D

解 因为 $AP_1 = B, P_2 B = E$, 所以 $B = P_2^{-1} = P_2$,

故 $AP_1 = P_2$, 于是 $A = P_2 P_1^{-1}$, 故应选 D.

(6) D

解 因为 $Ax = 0$ 的基础解系含一个线性无关的解向量, 所以 $r(A) = 3$, 于是 $r(A^*) = 1$,

故 $A^* x = 0$ 的基础解系含 3 个线性无关的解向量, 排除选项 A, B.

又 $A^* A = |A| E = 0$ 且 $r(A) = 3$, 所以 A 的列向量组中含有 $A^* x = 0$ 的基础解系,

因为 $(1, 0, 1, 0)^T$ 是方程组 $Ax = 0$ 的基础解系, 所以 $\alpha_1 + \alpha_3 = 0$,

故 $\alpha_1, \alpha_2, \alpha_4$ 或 $\alpha_2, \alpha_3, \alpha_4$ 线性无关, 显然 D 正确.

(7) D

解 $\because f_1(x)F_2(x) + f_2(x)F_1(x) \geq 0$,

$$\text{而 } \int_{-\infty}^{+\infty} f_1(x)F_2(x)dx + \int_{-\infty}^{+\infty} f_2(x)F_1(x)dx$$

$$= \int_{-\infty}^{+\infty} F_2(x)dF_1(x) + \int_{-\infty}^{+\infty} F_1(x)dF_2(x)$$

$$= F_1(x)F_2(x) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} F_1(x)dF_2(x) + \int_{-\infty}^{+\infty} F_1(x)dF_2(x) = 1,$$

$\therefore f_1(x)F_2(x) + f_2(x)F_1(x)$ 满足概率密度的两条性质. 故应选 D.

(8) B

解 因为 $UV = \begin{cases} XY, & \text{当 } X \geq Y \text{ 时,} \\ YX, & \text{当 } X < Y \text{ 时,} \end{cases}$

所以 $UV = XY$, 于是 $E(UV) = E(XY) = EX \cdot EY$. 故应选 B.

二、填空题

(9) $\ln(1 + \sqrt{2})$

解 因为 $y'(x) = \tan x$.

$$\text{所以 } s = \int_0^{\frac{\pi}{4}} \sqrt{1 + [y'(x)]^2} dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x} dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x dx = \ln |\sec x + \tan x| \Big|_0^{\frac{\pi}{4}} = \ln(1 + \sqrt{2}).$$

(10) $e^{-x} \sin x$

解 由条件知: $P(x) = 1, Q(x) = e^{-x} \cos x$, 于是微分方程通解为

$$y = e^{-\int P(x) dx} \left(\int Q(x) e^{\int P(x) dx} dx + C \right) = e^{-\int 1 dx} \left(\int e^{-x} \cos x e^{\int 1 dx} dx + C \right)$$

$$= e^{-x} \left(\int \cos x dx + C \right) = e^{-x} (\sin x + C),$$

由 $y(0) = 0$ 得 $C = 0$, 因此所求特解为 $y = e^{-x} \sin x$.

(11) 4

解 因为 $F(x, y) = \int_0^{xy} \frac{\sin t}{1+t^2} dt$, 所以

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{\sin(xy)}{1+(xy)^2} \cdot y = \frac{y \sin(xy)}{1+x^2 y^2}, \\ \frac{\partial^2 F}{\partial x^2} &= y^2 \cdot \frac{(1+x^2 y^2) \cdot \cos(xy) - 2xy \cdot \sin(xy)}{(1+x^2 y^2)^2},\end{aligned}$$

$$\text{于是 } \left. \frac{\partial^2 F}{\partial x^2} \right|_{\substack{x=0 \\ y=2}} = 4.$$

(12) π

解 由题设条件知 $P = xz, Q = x, R = \frac{y^2}{2}$,

根据斯托克斯公式得

$$\begin{aligned}\oint_L xz dx + x dy + \frac{y^2}{2} dz &= \iint_{\Sigma} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dx dz + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \iint_{\Sigma} (y - 0) dy dz + (x - 0) dx dz + (1 - 0) dx dy \\ &= \iint_{\Sigma} y dy dz + x dx dz + dx dy,\end{aligned}$$

其中 Σ 是位于柱面 $x^2 + y^2 = 1$ 内的平面 $z = x + y$, 取上侧, 且

$$\begin{aligned}\iint_{\Sigma} y dy dz &= 0, \iint_{\Sigma} x dx dz = 0, \\ \iint_{\Sigma} dx dy &= \iint_{D_{xy}: x^2+y^2 \leq 1} 1 \cdot dx dy = \pi,\end{aligned}$$

其中 D_{xy} 是 Σ 在 xOy 平面上的投影.

$$\text{因此 } \oint_L xz dx + x dy + \frac{y^2}{2} dz = \pi.$$

(13) 1

$$\text{解 } \mathbf{A} = \begin{pmatrix} 1 & a & 1 \\ a & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, f = \mathbf{x}^T \mathbf{A} \mathbf{x},$$

因为二次型经过正交变换化为 $y_1^2 + 4z_1^2 = 4$, 所以 $\mathbf{A}$ 的特征值为 $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 4$, 再由 $|\mathbf{A}| = -(a-1)^2 = \lambda_1 \lambda_2 \lambda_3 = 0$, 得 $a = 1$.

(14) $\mu\sigma^2 + \mu^3$

解: 因为 $(X, Y) \sim N(\mu, \mu; \sigma^2, \sigma^2; 0)$,

所以 $X \sim N(\mu, \sigma^2), Y \sim N(\mu, \sigma^2)$ 且 X, Y 相互独立.

$$\text{则 } E(XY^2) = EX \cdot E(Y^2) = EX \cdot [DY + (EY)^2] = \mu(\mu^2 + \sigma^2) = \mu\sigma^2 + \mu^3.$$

三、解答题

(15) 解 令 $y = \left(\frac{\ln(1+x)}{x} \right)^{\frac{1}{e^x-1}}$, 则当 $x > 0$ 时,

$$\ln y = \frac{1}{e^x - 1} \cdot \ln \left(\frac{\ln(1+x)}{x} \right) = \frac{\ln(\ln(1+x)) - \ln x}{e^x - 1},$$

$$\begin{aligned}
\text{而 } \lim_{x \rightarrow 0^+} (\ln y) &= \lim_{x \rightarrow 0^+} \frac{\ln(\ln(1+x)) - \ln x}{e^x - 1} = \lim_{x \rightarrow 0^+} \frac{\ln(\ln(1+x)) - \ln x}{x} \\
&= \lim_{x \rightarrow 0^+} \frac{\frac{1}{\ln(1+x)} \cdot \frac{1}{1+x} - \frac{1}{x}}{1} = \lim_{x \rightarrow 0^+} \frac{x - (1+x) \cdot \ln(1+x)}{x(1+x) \cdot \ln(1+x)} \\
&= \lim_{x \rightarrow 0^+} \frac{x - (1+x) \cdot \ln(1+x)}{x(1+x) \cdot x} = \lim_{x \rightarrow 0^+} \frac{1 - \ln(1+x) - 1}{2x(1+x) + x^2} \\
&= \lim_{x \rightarrow 0^+} \frac{-\frac{1}{1+x}}{2+6x} = -\frac{1}{2}.
\end{aligned}$$

当 $x < 0$ 时, $\ln y = \frac{\ln[-\ln(1+x)] - \ln(-x)}{e^x - 1}$, 同样可得

$$\lim_{x \rightarrow 0^-} \ln y = -\frac{1}{2}.$$

故 $\lim_{x \rightarrow 0} \left(\frac{\ln(1+x)}{x} \right)^{\frac{1}{e^x - 1}} = e^{-\frac{1}{2}}.$

(16) 解 由题设条件得 $g'(1) = 0, g(1) = 1$.

则 $\frac{\partial z}{\partial x} = y \cdot f'_1 + y g'(x) \cdot f'_2,$

$$\frac{\partial^2 z}{\partial x \partial y} = f'_1 + x y f''_{11} + y f''_{12} [g(x) + x g'(x)] + g'(x) \cdot f'_2 + y \cdot g(x) \cdot g'(x) \cdot f''_{22},$$

将 $x = 1, y = 1, g'(1) = 0, g(1) = 1$ 代入上式得

$$\left. \frac{\partial^2 z}{\partial x \partial y} \right|_{x=1, y=1} = f'_1(1, 1) + f''_{11}(1, 1) + f''_{12}(1, 1).$$

(17) 解 令 $f(x) = k \arctan x - x$, 则

$$f'(x) = \frac{k}{1+x^2} - 1 = \frac{k-1-x^2}{1+x^2}.$$

1) 当 $k \leq 1$ 时, $f'(x) = \frac{-(1-k)+x^2}{1+x^2} \leq 0$, 因此, $f(x)$ 单调减少,

此时 $f(x)$ 只有一个零点, 即 $f(0) = 0$, 即原方程 $k \arctan x - x = 0$ 只有一个实根 $x = 0$;

2) 当 $k > 1$ 时, 由 $f'(x) = 0$ 得 $x_1 = -\sqrt{k-1}, x_2 = \sqrt{k-1}$,

当 $x \in (-\infty, -\sqrt{k-1})$ 时, $f'(x) < 0$, 因此, $f(x)$ 单调减少;

当 $x \in (-\sqrt{k-1}, \sqrt{k-1})$ 时, $f'(x) > 0$, 因此, $f(x)$ 单调增加;

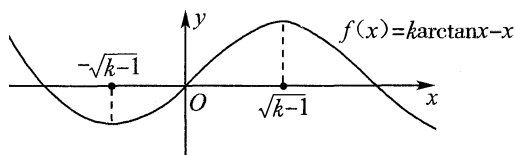
当 $x \in (\sqrt{k-1}, +\infty)$ 时, $f'(x) < 0$, 因此, $f(x)$ 单调减少,

所以 $x_1 = -\sqrt{k-1}$ 是极小值点, $x_2 = \sqrt{k-1}$ 是极大值点;

由于 $f(0) = 0$, 则 $f(x)$ 的极大值 $f(\sqrt{k-1}) > 0$, $f(x)$ 的极小值 $f(-\sqrt{k-1}) < 0$.

又 $\lim_{x \rightarrow -\infty} f(x) = +\infty, \lim_{x \rightarrow +\infty} f(x) = -\infty, f(0) = 0$,

综上知: $f(x)$ 在 $k > 1$ 时在 3 个不同的零点且分别位于 $(-\infty, -\sqrt{k-1})$, $(-\sqrt{k-1}, \sqrt{k-1})$ 及 $(\sqrt{k-1}, +\infty)$ 内, 此时函数 $f(x)$ 的草图如右图所示, 即原方程在 $k > 1$ 时有三个不同的实根.



(18) (I) 证 利用拉格朗日微分中值定理

令 $f(x) = \ln(1+x)$, 则 $f(x)$ 在闭区间 $\left[0, \frac{1}{n}\right)$ 上满足拉格朗日中值定理的条件, 于是有

$$f\left(\frac{1}{n}\right) - f(0) = f'(\xi) \cdot \frac{1}{n} \left(0 < \xi < \frac{1}{n}\right),$$

$$\text{即 } \ln\left(1 + \frac{1}{n}\right) = \ln\left(1 + \frac{1}{n}\right) - \ln 1 = \frac{1}{(1+\xi)n},$$

$$\because 0 < \xi < \frac{1}{n}, \therefore \frac{1}{1 + \frac{1}{n}} < \frac{1}{1 + \xi} < 1,$$

$$\text{则 } \frac{1}{1+n} < \frac{1}{n(1+\xi)} < \frac{1}{n},$$

$$\text{故有 } \frac{1}{1+n} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}.$$

$$\text{(II) 由(I)的结论知 } \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n},$$

$$\text{因此有 } \ln(n+1) - \ln n < \frac{1}{n},$$

令 $n = 1, 2, 3, \dots, n$ 得

$$\ln 2 - \ln 1 < 1,$$

$$\ln 3 - \ln 2 < \frac{1}{2},$$

$$\ln 4 - \ln 3 < \frac{1}{3},$$

.....

$$\ln(n+1) - \ln n < \frac{1}{n},$$

将上述各不等式两端分别相加得

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n},$$

$$\text{于是 } a_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \frac{1}{n+1} - \ln(n+1) > \frac{1}{n+1} > 0,$$

即数列 $\{a_n\}$ 有下界.

$$\text{又因为 } a_n - a_{n+1} = -\frac{1}{n+1} + \ln(n+1) - \ln n = \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} > 0 \text{ (由(I)的结论)}$$

即数列 $\{a_n\}$ 是单调下降的.

综上知数列 $\{a_n\}$ 单调下降且有界.

根据极限存在准则知 $\lim_{n \rightarrow \infty} a_n$ 存在且有限, 故数列 $\{a_n\}$ 收敛.

(19) 解 由题设条件知积分区域 D 可表示为: $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$,

于是有

$$\begin{aligned} I &= \iint_D xy f''_{xy}(x, y) dx dy \\ &= \int_0^1 x dx \int_0^1 y f''_{xy}(x, y) dy = \int_0^1 x dx \left[\int_0^1 y df'_x(x, y) \right] \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 x dx \left[(y f'_x(x, y)) \Big|_0^1 - \int_0^1 f'_x(x, y) dy \right] \\
&= \int_0^1 x dx \left[f'_x(x, 1) - \int_0^1 f'_x(x, y) dy \right] \\
&= \int_0^1 x f'_x(x, 1) dx - \int_0^1 x dx \int_0^1 f'_x(x, y) dy \\
&= \int_0^1 x df(x, 1) - \int_0^1 dy \int_0^1 x f'_x(x, y) dx \\
&= (x f(x, 1)) \Big|_0^1 - \int_0^1 f(x, 1) dx - \int_0^1 dy \int_0^1 x df(x, y) \\
&= - \int_0^1 dy \int_0^1 x df(x, y) \quad (\because f(x, 1) = 0) \\
&= - \int_0^1 dy \left[(x f(x, y)) \Big|_0^1 - \int_0^1 f(x, y) dx \right] \\
&= - \int_0^1 \left[f(1, y) - \int_0^1 f(x, y) dx \right] dy \\
&= \int_0^1 dy \int_0^1 f(x, y) dx \quad (\because f(1, y) = 0) \\
&= \iint_D f(x, y) dx dy = a,
\end{aligned}$$

故 $I = \iint_D xy f''_{xy}(x, y) dx dy = a.$

(20) 解 (I) 因为 $|\alpha_1, \alpha_2, \alpha_3| = \begin{vmatrix} 1 & 0 & 1 \\ & \cdots & \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{vmatrix} = 1 \neq 0,$

而 $\alpha_1, \alpha_2, \alpha_3$ 不能由 $\beta_1, \beta_2, \beta_3$ 线性表示, 故 $\beta_1, \beta_2, \beta_3$ 的秩小于 $\alpha_1, \alpha_2, \alpha_3$ 的秩, 从而

$$|\beta_1, \beta_2, \beta_3| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{vmatrix} = a - 5 = 0,$$

解得 $a = 5$.

(II) 解 设 $(\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_2, \alpha_3)C,$

$$\begin{aligned}
\text{则 } C &= (\alpha_1, \alpha_2, \alpha_3)^{-1} (\beta_1, \beta_2, \beta_3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 1 & -1 \\ 3 & 4 & -3 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 2 & 10 \\ -1 & 0 & -2 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\text{从而 } (\beta_1, \beta_2, \beta_3) &= (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} 2 & 1 & 5 \\ 4 & 2 & 10 \\ -1 & 0 & -2 \end{pmatrix} \\
&= (2\alpha_1 + 4\alpha_2 - \alpha_3, \alpha_1 + 2\alpha_2, 5\alpha_1 + 10\alpha_2 - 2\alpha_3),
\end{aligned}$$

即
$$\begin{cases} \beta_1 = 2\alpha_1 + 4\alpha_2 - \alpha_3, \\ \beta_2 = \alpha_1 + 2\alpha_2, \\ \beta_3 = 5\alpha_1 + 10\alpha_2 - 2\alpha_3. \end{cases}$$

(21) 解 (I) 记 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, 由题设条件知 $A\alpha_1 = -\alpha_1, A\alpha_2 = \alpha_2$.

所以, A 有特征值 $\lambda_1 = -1, \lambda_2 = 1, \alpha_1, \alpha_2$ 为其对应的特征向量.

又秩(A) = 2, 故 $|A| = 0$, 从而, 另一特征值 $\lambda_3 = 0$,

设 $\lambda_3 = 0$ 对应的特征向量 $\alpha_3 = (x_1, x_2, x_3)^T$, 由于 A 为实对称矩阵, 则 $\begin{cases} \alpha_1^T \alpha_3 = 0, \\ \alpha_2^T \alpha_3 = 0, \end{cases}$

即 $\begin{cases} x_1 - x_3 = 0, \\ x_1 + x_3 = 0. \end{cases}$

解得 $\alpha_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

从而 A 的特征值分别为 $-1, 1, 0$, 其对应的特征向量分别为 $k_1\alpha_1, k_2\alpha_2, k_3\alpha_3$ (其中 $k_i \neq 0, i = 1, 2, 3$).

(II) 由于不同特征值的特征向量正交, 则只需将 $\alpha_1, \alpha_2, \alpha_3$ 单位化, 得

$$r_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, r_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, r_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\text{令 } Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \text{ 则 } Q^T A Q = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix},$$

$$\begin{aligned} \text{所以 } A &= Q \begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix} Q^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

(22) 解 (I) 由 $P\{X^2 = Y^2\} = 1$ 可知 $P\{X^2 \neq Y^2\} = 0$,

于是 $P\{X=0, Y=1\} = P\{X=0, Y=-1\} = P\{X=1, Y=0\} = 0$,

则 $P\{X=1, Y=-1\} = P\{Y=-1\} - P\{X=0, Y=-1\} = \frac{1}{3}$,

同理 $P\{X=1, Y=1\} = P\{X=0, Y=0\} = \frac{1}{3}$,

即概率分布如下

$X \backslash Y$	-1	0	1
0	0	$\frac{1}{3}$	0
1	$\frac{1}{3}$	0	$\frac{1}{3}$

(II) $Z = XY$ 的可能取值为 $-1, 0, 1$.

$$P\{XY = -1\} = P\{X = 1, Y = -1\} = \frac{1}{3},$$

$$P\{XY = 1\} = P\{X = 1, Y = 1\} = \frac{1}{3},$$

$$P\{XY = 0\} = 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3},$$

故 Z 的分布律为

Z	-1	0	1
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$(III) EX = \frac{2}{3}, EY = 0, E(XY) = 0,$$

$$\text{故 } \text{Cov}(X, Y) = E(XY) - EX \cdot EY = 0,$$

$$\text{则 } \rho_{XY} = \frac{\text{Cov}(X, Y)}{\sqrt{DX} \sqrt{DY}} = 0.$$

(23) 解 (I) 设 $x_1, x_2, \dots, x_n$ 为样本观测值, 则似然函数为

$$L(\sigma^2) = \prod_{i=1}^n f(x_i) = \frac{1}{(\sqrt{2\pi})^n \sigma^n} e^{-\frac{\sum_{i=1}^n (x_i - \mu_0)^2}{2\sigma^2}},$$

$$\text{则 } \ln L(\sigma^2) = -n \ln \sqrt{2\pi} - \frac{n}{2} \ln \sigma^2 - \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{2\sigma^2},$$

$$\text{令 } \frac{d \ln L(\sigma^2)}{d \sigma^2} = -\frac{n}{2\sigma^2} + \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{2\sigma^4} = 0,$$

$$\text{得 } \sigma^2 \text{ 的极大似然估计为 } \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu_0)^2.$$

$$(II) \text{ 因为 } \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu_0)^2 \sim \chi^2(n),$$

$$\text{所以 } E\left[\frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu_0)^2\right] = n, D\left[\frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu_0)^2\right] = 2n,$$

$$\text{于是 } E(\hat{\sigma}^2) = \frac{\sigma^2}{n} E\left[\frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu_0)^2\right] = \sigma^2,$$

$$D(\hat{\sigma}^2) = \frac{\sigma^4}{n^2} D\left[\frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu_0)^2\right] = \frac{2}{n} \sigma^4.$$