

2010年(数一) 真题答案解析

一、选择题

(1) C

解法 用求幂指数型极限的一般方法。求 $I = \lim_{x \rightarrow \infty} e^{x \ln \frac{x^2}{(x-a)(x+b)}}$,

$$\begin{aligned} \text{归结为求 } W &= \lim_{x \rightarrow \infty} x \ln \frac{x^2}{(x-a)(x+b)} = \lim_{x \rightarrow \infty} x \ln \left(\frac{x^2}{(x-a)(x+b)} - 1 + 1 \right) \\ &= \lim_{x \rightarrow \infty} x \left(\frac{x^2}{(x-a)(x+b)} - 1 \right) = \lim_{x \rightarrow \infty} \left(x \cdot \frac{(a-b)x + ab}{(x-a)(x+b)} \right) = a - b. \end{aligned}$$

因此 $I = e^{a-b}$. 故应选 C.

(2) B

解 因为 $z = z(x, y)$ 由方程 $F\left(\frac{y}{x}, \frac{z}{x}\right) = 0$ 确定, 则对 $F\left(\frac{y}{x}, \frac{z}{x}\right)$ 求偏导数得

$$F'_x = F'_1 \left(-\frac{y}{x^2}\right) + F'_2 \left(-\frac{z}{x^2}\right), \quad F'_y = F'_1 \cdot \frac{1}{x}, \quad F'_z = F'_2 \cdot \frac{1}{x},$$

$$\text{所以 } \frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} = -\frac{F'_1 \left(-\frac{y}{x^2}\right) + F'_2 \left(-\frac{z}{x^2}\right)}{F'_2 \cdot \frac{1}{x}} = \frac{yF'_1 + zF'_2}{xF'_2},$$

$$\frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z} = -\frac{F'_1 \frac{1}{x}}{F'_2 \frac{1}{x}} = -\frac{F'_1}{F'_2},$$

$$\text{则 } x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{yF'_1 + zF'_2}{F'_2} - \frac{yF'_1}{F'_2} = z.$$

(3) D

解 显然广义积分 $\int_0^1 \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx$ 有两个瑕点 $x=0$ 与 $x=1$, 则

$$\int_0^1 \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx = \int_0^{\frac{1}{2}} \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx + \int_{\frac{1}{2}}^1 \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx,$$

对于 $\int_0^{\frac{1}{2}} \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx$, 瑕点为 $x=0$,

设 $n > 1$, $\lim_{x \rightarrow 0^+} \frac{[\ln^2(1-x)]^{\frac{1}{m}}}{x^{\frac{1}{n}}} \cdot x^{\frac{1}{n}} = 0$, 由于 $0 < \frac{1}{n} < 1$, 故收敛.

设 $n=1, m=1, 2$, $\lim_{x \rightarrow 0^+} \frac{[\ln^2(1-x)]^{\frac{1}{m}}}{x^{\frac{1}{n}}}$ 存在, 故此时不是反常积分.

设 $n=1, m > 2$, $\lim_{x \rightarrow 0^+} \frac{[\ln^2(1-x)]^{\frac{1}{m}}}{x} \cdot x^{1-\frac{2}{m}}$ 存在, 又 $0 < 1 - \frac{2}{m} < 1$, 故收敛.

对于 $\int_{\frac{1}{2}}^1 \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx$, 瑕点为 $x=1$, 当 m 为正整数时, $\lim_{x \rightarrow 1^-} \frac{[\ln^2(1-x)]^{\frac{1}{m}}}{x^{\frac{1}{n}}} \cdot (1-x)^{\frac{1}{2}} = 0$,

故收敛.

所以, 不论 m, n 取何正整数, 反常积分都收敛. 故选 D.

(4) D

$$\begin{aligned} \text{解} \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n \frac{n}{(n+i)(n^2+j^2)} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n \frac{n}{n \left(1 + \frac{i}{n}\right) \cdot n^2 \left(1 + \left(\frac{j}{n}\right)^2\right)} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n \frac{1}{n^2} \cdot \frac{1}{1 + \frac{i}{n}} \cdot \frac{1}{1 + \left(\frac{j}{n}\right)^2} = \int_0^1 dx \int_0^1 \frac{1}{(1+x)(1+y^2)} dy. \end{aligned}$$

(5) A

解 由于 A 为 $m \times n$ 矩阵, B 为 $n \times m$ 矩阵, 故 $r(A) \leq m, r(B) \leq m$.

又 $AB = E$, 于是 $m = r(AB) \leq r(A) \leq m, m = r(AB) \leq r(B) \leq m$,

所以 $r(A) = m, r(B) = m$.

(6) D

解 设 λ 是 A 的特征值. 由于 $A^2 + A = 0$,

所以 $\lambda^2 + \lambda = 0$, 即 $(\lambda + 1)\lambda = 0$,

故 A 的特征值为 -1 或 0 . 又 A 为实对称矩阵, 所以 A 可相似于对角阵 Λ .

且 $r(A) = r(\Lambda) = 3$,

$$\text{于是 } \Lambda = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}.$$

(7) C

$$\text{解} \quad P\{X=1\} = P\{X \leq 1\} - P\{X < 1\} = F(1) - F(1-0) = 1 - e^{-1} - \frac{1}{2} = \frac{1}{2} - e^{-1}.$$

(8) A

$$\text{解} \quad \text{由于 } f_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

$$f_2(x) = \begin{cases} \frac{1}{4}, & 1 \leq x \leq 3, \\ 0, & \text{其他.} \end{cases}$$

$$\text{故 } \int_{-\infty}^{+\infty} f(x) dx = a \int_{-\infty}^0 f_1(x) dx + b \int_0^{+\infty} f_2(x) dx = a \times \frac{1}{2} + b \int_0^3 \frac{1}{4} dx = \frac{a}{2} + \frac{3}{4}b = 1. \text{ 可得}$$

$$2a + 3b = 4, \text{ 故应选 A.}$$

二、填空题

(9) 0

解 由题设条件 $x'(t) = -e^{-t}, y'(t) = \ln(1+t^2)$,

$$\text{则 } \frac{dy}{dx} = \frac{y'(t)}{x'(t)} = -\ln(1+t^2)e^t,$$

$$\begin{aligned}\frac{d^2 y}{dx^2} &= \frac{dy}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{1}{x'(t)} = \left(-e^t \ln(1+t^2) - e^t \cdot \frac{2t}{1+t^2} \right) (-e^t) \\ &= e^{2t} \ln(1+t^2) + \frac{2te^{2t}}{1+t^2} = e^{2t} \left[\ln(1+t^2) + \frac{2t}{1+t^2} \right].\end{aligned}$$

从而 $\frac{d^2 y}{dx^2} \Big|_{t=0} = 0$, 故应填 0.

(10) -4π

解 $\int_0^{\pi^2} \sqrt{x} \cos \sqrt{x} dx$

$$\begin{aligned}&\stackrel{\text{令 } t = \sqrt{x}}{=} \int_0^{\pi} t \cos t \cdot 2t dt = 2 \int_0^{\pi} t^2 \cos t dt = 2 \left[(t^2 \cdot \sin t) \Big|_0^{\pi} - \int_0^{\pi} \sin t \cdot 2t dt \right] \\ &= -4 \int_0^{\pi} t \cdot \sin t dt = 4 \left[(t \cdot \cos t) \Big|_0^{\pi} - \int_0^{\pi} \cos t dt \right] = 4(-\pi - 0) - 4 \sin t \Big|_0^{\pi} = -4\pi.\end{aligned}$$

(11) 0

解 利用格林公式如右图所示, 由题设条件知.

$$L = \overline{AB} + \overline{BC}.$$

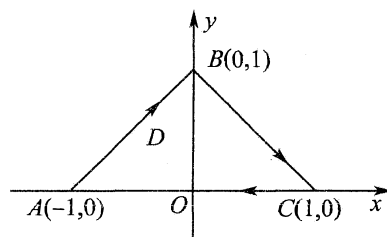
记 $L_1 = \overline{CA}$, 则 $L + L_1$ 为闭曲线且所围区域记为 D ,

此时, $P(x, y) = xy$, $Q(x, y) = x^2$, 且 $\frac{\partial Q}{\partial x} = 2x$, $\frac{\partial P}{\partial y} = x$.

由格林公式知

$$\oint_{-(L+L_1)} xy dx + x^2 dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_D x dx dy = 0 \text{ (由对称性),}$$

$$\begin{aligned}\text{则 } \int_L xy dx + x^2 dy &= - \int_{-L} xy dx + x^2 dy \\ &= - \oint_{-(L+L_1)} xy dx + x^2 dy + \int_{-L_1} xy dx + x^2 dy \\ &= \int_{-L_1} xy dx + x^2 dy = \int_{-1}^1 (x \cdot 0 + x^2 \cdot 0) dx = 0.\end{aligned}$$



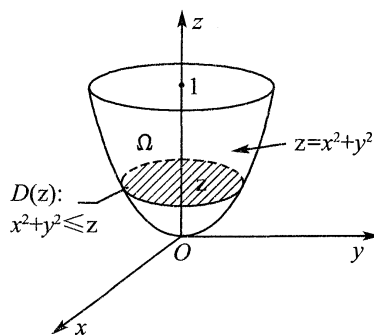
(12) $\frac{2}{3}$

解 由题设所求坐标为 $\bar{z} = \frac{\iiint_{\Omega} z dV}{\iiint_{\Omega} dV}$,

其中积分区域 Ω 如右图所示用平面 $z = z (0 \leq z \leq 1)$ 截积分区域 Ω 得截面 $D(z)$ 且 $D(z)$ 是一圆域: $x^2 + y^2 \leq z$.

$$\begin{aligned}\text{于是 } \iiint_{\Omega} z dV &= \int_0^1 dz \iint_{D(z)} z dx dy = \int_0^1 z dz \iint_{D(z)} dx dy \\ &= \int_0^1 z \cdot \pi z dz = \pi \int_0^1 z^2 dz = \frac{\pi}{3} z^3 \Big|_0^1 = \frac{\pi}{3},\end{aligned}$$

$$\iiint_{\Omega} dV = \int_0^1 dz \iint_{D(z)} dx dy = \int_0^1 \pi z dz = \frac{\pi}{2} z^2 \Big|_0^1 = \frac{\pi}{2},$$



即 $\bar{z} = \frac{\pi}{3} \cdot \frac{2}{\pi} = \frac{2}{3}$, 故应填 $\frac{2}{3}$.

(13) 6

解 由于 $\alpha_1, \alpha_2, \alpha_3$ 生成的向量空间的维数为 2, 所以 $r(\alpha_1, \alpha_2, \alpha_3) = 2$.

对矩阵 $(\alpha_1, \alpha_2, \alpha_3)$ 进行初等行变换:

$$(\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & a \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \\ 0 & 2 & a \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & a-6 \\ 0 & 0 & 0 \end{pmatrix},$$

所以 $a = 6$. 故应填 6.

(14) 2

解 因为 $\sum_{k=0}^{\infty} P_k = 1$, 故

$$\sum_{k=0}^{\infty} P_k = \sum_{k=0}^{\infty} \frac{C}{k!} = C \sum_{k=0}^{\infty} \frac{1}{k!} = Ce = 1 \quad (\text{其中 } \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^\lambda).$$

即得 $C = e^{-1}$. 所以

$$P\{X=k\} = \frac{e^{-1}}{k!}, k=0, 1, 2, \dots$$

$$\text{则 } EX^2 = \sum_{k=0}^{\infty} k^2 \cdot \frac{e^{-1}}{k!} = e^{-1} \cdot \sum_{k=1}^{\infty} \frac{k}{(k-1)!} = e^{-1} \sum_{k=1}^{\infty} \frac{(k-1)+1}{(k-1)!} = 2.$$

三、解答题

(15) 解 由题设知, 齐次方程对应的特征方程为 $r^2 - 3r + 2 = 0$,

解得特征根为: $r_1 = 1, r_2 = 2$.

于是齐次方程 $y'' - 3y' + 2y = 0$ 的通解是:

$$Y = C_1 e^x + C_2 e^{2x}, (C_1, C_2 \text{ 是任意常数}).$$

由条件知原方程的一个特解可设为: $y_1 = x(ax+b)e^x$, (其中 a, b 为待定系数).

$$\text{则 } y_1' = [ax^2 + (2a+b) \cdot x + b]e^x, y_1'' = [ax^2 + (4a+b)x + 2a + 2b]e^x.$$

将 y_1, y_1', y_1'' 代入原方程并整理得

$$y_1'' - 3y_1' + 2y_1 = (2a - b - 2ax)e^x = 2xe^x$$

比较等式两端 x 同次幂的系数得

$$\begin{cases} -2a = 2 \\ 2a - b = 0 \end{cases}, \text{即} \begin{cases} a = -1 \\ b = -2 \end{cases}$$

于是特解 $y_1 = -x(x+2)e^x$.

故原方程通解为

$$y = Y + y_1 = C_1 e^x + C_2 e^{2x} - x(x+2)e^x \quad (\text{其中 } C_1, C_2 \text{ 是任意常数}).$$

(16) 解 由 $f(x) = \int_1^{x^2} (x^2 - t)e^{-t^2} dt = x^2 \int_1^{x^2} e^{-t^2} dt - \int_1^{x^2} t e^{-t^2} dt$,

$$\text{则 } f'(x) = 2x \int_1^{x^2} e^{-t^2} dt + x^2 e^{-(x^2)^2} \cdot 2x - x^2 e^{-(x^2)^2} 2x = 2x \int_1^{x^2} e^{-t^2} dt.$$

$$f''(x) = 2 \int_1^{x^2} e^{-t^2} dt + 2x e^{-(x^2)^2} \cdot 2x = 2 \int_1^{x^2} e^{-t^2} dt + 4x^2 e^{-x^4}.$$

令 $f'(x) = 0$, 得 $x = 0, x = \pm 1$.

则 $f''(0) = 2 \int_1^0 e^{-t^2} dt < 0$, $f''(\pm 1) = 4e^{-1} > 0$,

所以 $f(0) = \frac{1}{2}(1 - e^{-1})$ 是极大值, $f(\pm 1) = 0$ 是极小值.

由于当 $x > 1$ 时, $f'(x) > 0$; $0 < x < 1$ 时, $f'(x) < 0$; $-1 < x < 0$ 时, $f'(x) > 0$; $x < -1$ 时, $f'(x) < 0$.

故 $f(x)$ 的单调递减区间为 $(-\infty, -1) \cup (0, 1)$,

$f(x)$ 的单调递增区间为 $(-1, 0) \cup (1, +\infty)$.

(17) 解 (I) 当 $0 \leq x \leq 1$ 时, $0 \leq \ln(1+x) \leq x$,

故当 $0 \leq t \leq 1$ 时, $[\ln(1+t)]^n \leq t^n$, 所以 $|\ln t| [\ln(1+t)]^n \leq t^n |\ln t|$.

所以 $\int_0^1 |\ln t| [\ln(1+t)]^n dt \leq \int_0^1 t^n |\ln t| dt$.

(II) 由(I)知 $0 \leq u_n \leq \int_0^1 |\ln t| t^n dt = \frac{1}{(1+n)^2}$,

而 $\int_0^1 |\ln t| t^n dt = -\int_0^1 t^n \ln t dt = -\frac{1}{n+1} t^{n+1} \ln t \Big|_0^1 + \int_0^1 \frac{1}{n+1} t^n dt = \frac{1}{(n+1)^2}$,

又由于 $\lim_{n \rightarrow \infty} \frac{1}{(1+n)^2} = 0$,

根据夹逼准则知, $\lim_{n \rightarrow \infty} u_n = 0$.

(18) 解 ① 记 $u_n(x) = \frac{(-1)^{n-1}}{2n-1} x^{2n}$,

因为 $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n x^{2n+2}}{2n+1} \cdot \frac{2n-1}{(-1)^{n-1} \cdot x^{2n}} \right| = x^2$,

所以由比值法知, 当 $x^2 < 1$ 即 $|x| < 1$ 时, 级数收敛; 当 $x^2 > 1$ 即 $|x| > 1$ 时, 级数发散. 于是可知幂级数的收敛半径 $R = 1$, 即收敛区间为 $(-1, 1)$;

当 $x = \pm 1$ 时, 级数 $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ 为交错级数, 由莱布尼茨定理知级数收敛,

故幂级数 $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n}$ 的收敛域为 $[-1, 1]$.

② 记 $S(x)$ 为级数 $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n}$ 的和函数, 则

$$S(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n} = x \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1} = x \cdot S_1(x)$$

其中 $S_1(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1}$, $x \in [-1, 1]$

由幂级数和函数的性质得

$$\begin{aligned} S_1'(x) &= \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1} \right)' = \sum_{n=1}^{\infty} (-1)^{n-1} x^{2n-2} \\ &= 1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1} x^{2n-2} + \cdots \\ &= \frac{1}{1+x^2}, x \in [-1, 1]. \end{aligned}$$

所以 $S_1(x) = \int_0^x S_1'(t) dt + S_1(0) = \int_0^x \frac{1}{1+t^2} dt + 0 = \arctan t \Big|_0^x = \arctan x$.

故 $S(x) = xS_1(x) = x \arctan x, x \in [-1, 1]$.

(19) 解 ① 令 $F(x, y, z) = x^2 + y^2 + z^2 - yz - 1$, 则 $F(x, y, z) = 0$ 为椭球面 S 的方程, 设点 P 的坐标为 (x, y, z) , 由题设条件知曲面 S 在点 P 处的切平面法向量为:

$$\mathbf{n}_1 = \{F'_x, F'_y, F'_z\} = \{2x, 2y - z, 2z - y\},$$

又 xOy 平面的法向量为: $\mathbf{n}_2 = \{0, 0, 1\}$, 由于点 P 处的切平面垂直于 xOy 平面, 于是

$$\mathbf{n}_1 \perp \mathbf{n}_2 \Leftrightarrow \mathbf{n}_1 \cdot \mathbf{n}_2 = 0, \text{ 即 } y = 2z.$$

又因为点 P 在曲面 S 上, 所以点 P 的坐标 (x, y, z) 满足曲面 S 的方程: $x^2 + y^2 + z^2 - yz = 1$, 从而知动点 P 的轨迹 C 的方程为

$$\begin{cases} x^2 + y^2 + z^2 - yz = 1 \\ y = 2z \end{cases}$$

② 根据题设条件知, 曲面积分 $\iint_{\Sigma} \frac{(x + \sqrt{3}) |y - 2z|}{\sqrt{4 + y^2 + z^2 - 4yz}} dS$ 中积分曲面 Σ 是椭球面 S 位于平面

$y = 2z$ 上方的部分, 因此在 Σ 上: $y \leq 2z$, 于是 $|y - 2z| = 2z - y$, 即

$$\iint_{\Sigma} \frac{(x + \sqrt{3}) |y - 2z|}{\sqrt{4 + y^2 + z^2 - 4yz}} dS = \iint_{\Sigma} \frac{(x + \sqrt{3})(2z - y)}{\sqrt{4 + y^2 + z^2 - 4yz}} dS$$

在曲面 Σ 的方程: $x^2 + y^2 + z^2 - yz = 1$ 两端分别对 x, y 求偏导数(此时, $z = z(x, y)$) 得

$$2x + 2z \cdot \frac{\partial z}{\partial x} - y \cdot \frac{\partial z}{\partial x} = 0, \text{ 即 } \frac{\partial z}{\partial x} = \frac{2x}{y - 2z}$$

$$2y + 2z \cdot \frac{\partial z}{\partial y} - z - y \frac{\partial z}{\partial y} = 0, \text{ 即 } \frac{\partial z}{\partial y} = \frac{2y - z}{y - 2z}$$

将曲面 Σ 向 xOy 面投影, 得投影域为:

$$D_{xy}: x^2 + \frac{3}{4}y^2 \leq 1.$$

又因为 $dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy$

$$= \frac{\sqrt{4x^2 + 5y^2 + 5z^2 - 8yz}}{2z - y} dx dy$$

$$= \frac{\sqrt{4(x^2 + y^2 + z^2 - yz) + y^2 + z^2 - 4yz}}{2z - y} dx dy$$

$$= \frac{\sqrt{4 + y^2 + z^2 - 4yz}}{2z - y} dx dy \quad (\text{因 } x^2 + y^2 + z^2 - yz = 1)$$

$$\text{所以 } I = \iint_{\Sigma} \frac{(x + \sqrt{3})(2z - y)}{\sqrt{4 + y^2 + z^2 - 4yz}} dS = \iint_{D_{xy}} \frac{(x + \sqrt{3})(2z - y)}{\sqrt{4 + y^2 + z^2 - 4yz}} \cdot \frac{\sqrt{4 + y^2 + z^2 - 4yz}}{2z - y} dx dy$$

$$= \iint_{D_{xy}} (x + \sqrt{3}) dx dy = \iint_{D_{xy}} x dx dy + \sqrt{3} \iint_{D_{xy}} dx dy = 0 + \sqrt{3} \pi \cdot 1 \cdot \frac{2}{\sqrt{3}} = 2\pi.$$

(20) 解 (I) 已知 $Ax = b$ 有 2 个不同的解, 所以 $r(A) = r(A : b) < 3$.

$$\text{又 } |A| = 0, \text{ 即 } |A| = \begin{vmatrix} \lambda & 1 & 1 \\ 0 & \lambda - 1 & 0 \\ 1 & 1 & \lambda \end{vmatrix} = (\lambda - 1)^2(\lambda + 1) = 0, \text{ 知 } \lambda = 1 \text{ 或 } -1.$$

当 $\lambda = 1$ 时, $r(A) = 1 \neq r(A : b) = 2$, 此时 $Ax = b$ 无解, 故 $\lambda = -1$.

又由 $r(\mathbf{A}) = r(\mathbf{A} : \mathbf{b})$ 得 $a = -2$.

$$(II) \text{ 因 } (\mathbf{A} : \mathbf{b}) = \left(\begin{array}{ccc|c} -1 & 1 & 1 & -2 \\ 0 & -2 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array} \right)$$

则原方程组的等价方程组为 $\begin{cases} x_1 = \frac{3}{2} + x_3 \\ x_2 = -\frac{1}{2} \end{cases}$, 其中 x_3 为自由未知量.

$$\text{令 } x_3 = 0, \text{ 得方程组特解 } \mathbf{u}_0 = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}.$$

又方程组对应的齐次方程组的等价方程组为 $\begin{cases} x_1 = x_3 \\ x_2 = 0 \end{cases}$, 其中 x_3 为自由未知量.

$$\text{令 } x_3 = 1, \text{ 得齐次方程组 } \mathbf{Ax} = \mathbf{0} \text{ 的基础解系 } \boldsymbol{\alpha} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$\text{所以 } \mathbf{Ax} = \mathbf{b} \text{ 的通解为 } \mathbf{x} = k \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}, \text{ 其中 } k \text{ 为任意常数.}$$

(21) 解 (I) 由于二次型在正交变换 $\mathbf{x} = \mathbf{Qy}$ 下的标准形为 $y_1^2 + y_2^2$, 所以 $\mathbf{A}$ 的特征值为 $\lambda_1 = \lambda_2 = 1, \lambda_3 = 0$.

由于 $\mathbf{Q}$ 的第 3 列为 $\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)^T$, 所以 $\mathbf{A}$ 对应于 $\lambda_3 = 0$ 的特征向量为 $\boldsymbol{\alpha}_3 = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)^T$.

由于 $\mathbf{A}$ 是实对称矩阵, 所以对应于不同特征值的特征向量是相互正交的, 设属于 $\lambda_1 = \lambda_2 = 1$

的特征向量为 $\boldsymbol{\alpha} = (x_1, x_2, x_3)^T$, 则 $\boldsymbol{\alpha}^T \boldsymbol{\alpha}_3 = 0$, 即 $\frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_3 = 0$, 取

$$\boldsymbol{\alpha}_1 = (0, 1, 0)^T, \boldsymbol{\alpha}_2 = (-1, 0, 1)^T,$$

则 $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ 与 $\boldsymbol{\alpha}_3$ 是正交的, 即为对应于 $\lambda_1 = \lambda_2 = 1$ 的特征向量.

由于 $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ 是相互正交的, 所以只需单位化:

$$\boldsymbol{\beta}_1 = \frac{\boldsymbol{\alpha}_1}{\|\boldsymbol{\alpha}_1\|} = (0, 1, 0)^T, \boldsymbol{\beta}_2 = \frac{\boldsymbol{\alpha}_2}{\|\boldsymbol{\alpha}_2\|} = \frac{1}{\sqrt{2}}(-1, 0, 1)^T.$$

$$\text{取 } \mathbf{Q} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\alpha}_3) = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \text{ 则 } \mathbf{Q}^T \mathbf{A} \mathbf{Q} = \boldsymbol{\Lambda} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix},$$

从而 $A = Q\Lambda Q^T = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$

(II) 由于 A 的特征值为 $1, 1, 0$, 所以 $A + E$ 的特征值为 $2, 2, 1$, 则 $A + E$ 的特征值全大于零, 故 $A + E$ 是正定矩阵.

(22) 解 因为 $f(x, y) = A e^{-2x^2 + 2xy - y^2} = A e^{-(x-y)^2} \cdot e^{-x^2}$

$$= A \pi \left[\frac{1}{\sqrt{2\pi} \cdot \frac{1}{\sqrt{2}}} e^{-\frac{(x-y)^2}{2 \cdot (\frac{1}{\sqrt{2}})^2}} \right] \left[\frac{1}{\sqrt{2\pi} \left(\frac{1}{\sqrt{2}} \right)} e^{-\frac{x^2}{2 \cdot (\frac{1}{\sqrt{2}})^2}} \right]$$

由概率密度的性质得到

$$1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) dx dy = A \pi \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi} \left(\frac{1}{\sqrt{2}} \right)} e^{-\frac{x^2}{2 \cdot (\frac{1}{\sqrt{2}})^2}} dx \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi} \cdot \frac{1}{\sqrt{2}}} e^{-\frac{(x-y)^2}{2 \cdot (\frac{1}{\sqrt{2}})^2}} dy = A \pi$$

故 $A = \frac{1}{\pi}.$

从而 $f(x, y) = \frac{1}{\pi} e^{-2x^2 + 2xy - y^2} (-\infty < x < +\infty, -\infty < y < +\infty),$

又 $f_X(x) = \int_{-\infty}^{+\infty} f(x, y) dy = \frac{1}{\sqrt{\pi}} e^{-x^2} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi} \cdot \frac{1}{\sqrt{2}}} e^{-\frac{(x-y)^2}{2 \cdot (\frac{1}{\sqrt{2}})^2}} dy = \frac{1}{\sqrt{\pi}} e^{-x^2}.$

所以 $f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{1}{\sqrt{\pi}} e^{-x^2 + 2xy - y^2} = \frac{1}{\sqrt{\pi}} e^{-(x-y)^2} (-\infty < x < +\infty, -\infty < y < +\infty)$

(23) 解 因为 $N_1 \sim B(n, 1 - \theta), N_2 \sim B(n, \theta - \theta^2), N_3 \sim B(n, \theta^2),$

所以
$$\begin{aligned} ET &= E\left(\sum_{i=1}^3 a_i N_i\right) = a_1 EN_1 + a_2 EN_2 + a_3 EN_3 \\ &= a_1 n(1 - \theta) + a_2 n(\theta - \theta^2) + a_3 n\theta^2 \\ &= na_1 + n(a_2 - a_1)\theta + n(a_3 - a_2)\theta^2 \end{aligned}$$

由 T 是 θ 的无偏估计量, 可知 $ET = \theta,$

则
$$\begin{cases} na_1 = 0, \\ n(a_2 - a_1) = 1, \\ n(a_3 - a_2) = 0, \end{cases} \text{ 即 } \begin{cases} a_1 = 0, \\ a_2 = \frac{1}{n}, \\ a_3 = \frac{1}{n}. \end{cases}$$

故 $T = 0 \times N_1 + \frac{1}{n} \times N_2 + \frac{1}{n} \times N_3 = \frac{1}{n}(N_2 + N_3) = \frac{1}{n}(n - N_1).$

$$DT = D\left[\frac{1}{n}(n - N_1)\right] = \frac{1}{n^2} DN_1 = \frac{1}{n^2} \cdot n \cdot (1 - \theta) \cdot \theta = \frac{1}{n} \theta (1 - \theta).$$