

2010 年全国硕士研究生入学统一考试

数学二试题参考答案

一、选择题

(1) 【答案】 (B).

【解析】 因为 $f(x) = \frac{x^2 - x}{x^2 - 1} \sqrt{1 + \frac{1}{x^2}}$ 有间断点 $x = 0, \pm 1$, 又因为

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{x(x-1)}{(x+1)(x-1)} \sqrt{1 + \frac{1}{x^2}} = \lim_{x \rightarrow 0} x \sqrt{1 + \frac{1}{x^2}},$$

其中 $\lim_{x \rightarrow 0^+} x \sqrt{1 + \frac{1}{x^2}} = 1, \lim_{x \rightarrow 0^-} x \sqrt{1 + \frac{1}{x^2}} = -1$, 所以 $x = 0$ 为跳跃间断点.

显然 $\lim_{x \rightarrow 1} f(x) = \frac{1}{2} \sqrt{1+1} = \frac{\sqrt{2}}{2}$, 所以 $x = 1$ 为连续点.

而 $\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{x(x-1)}{(x+1)(x-1)} \sqrt{1 + \frac{1}{x^2}} = \infty$, 所以 $x = -1$ 为无穷间断点, 故答案选择

B.

(2) 【答案】 (A).

【解析】 因 $\lambda y_1 - \mu y_2$ 是 $y' + P(x)y = 0$ 的解, 故 $(\lambda y_1 - \mu y_2)' + P(x)(\lambda y_1 - \mu y_2) = 0$, 所以

$$\lambda [y_1' + P(x)y_1] - \mu [y_2' + P(x)y_2] = 0,$$

而由已知 $y_1' + P(x)y_1 = q(x), y_2' + P(x)y_2 = q(x)$, 所以

$$(\lambda - \mu)q(x) = 0, \quad \text{①}$$

又由于一阶次微分方程 $y' + p(x)y = q(x)$ 是非齐的, 由此可知 $q(x) \neq 0$, 所以

$$\lambda - \mu = 0.$$

由于 $\lambda y_1 + \mu y_2$ 是非齐次微分方程 $y' + P(x)y = q(x)$ 的解, 所以

$$(\lambda y_1 + \mu y_2)' + P(x)(\lambda y_1 + \mu y_2) = q(x),$$

整理得

$$\lambda [y_1' + P(x)y_1] + \mu [y_2' + P(x)y_2] = q(x),$$

即

$$(\lambda + \mu)q(x) = q(x), \text{ 由 } q(x) \neq 0 \text{ 可知 } \lambda + \mu = 1, \quad \text{②}$$

由①②求解得 $\lambda = \mu = \frac{1}{2}$, 故应选 (A).

(3) 【答案】 (C).

【解析】因为曲线 $y = x^2$ 与曲线 $y = a \ln x (a \neq 0)$ 相切, 所以在切点处两个曲线的斜率相同,

所以 $2x = \frac{a}{x}$, 即 $x = \sqrt{\frac{a}{2}} (x > 0)$. 又因为两个曲线在切点的坐标是相同的, 所以在 $y = x^2$ 上,

当 $x = \sqrt{\frac{a}{2}}$ 时 $y = \frac{a}{2}$; 在 $y = a \ln x$ 上, $x = \sqrt{\frac{a}{2}}$ 时, $y = a \ln \sqrt{\frac{a}{2}} = \frac{a}{2} \ln \frac{a}{2}$.

所以 $\frac{a}{2} = \frac{a}{2} \ln \frac{a}{2}$. 从而解得 $a = 2e$. 故答案选择 (C).

(4) 【答案】 (D).

【解析】 $x = 0$ 与 $x = 1$ 都是瑕点. 应分成

$$\int_0^1 \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx = \int_0^{\frac{1}{2}} \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx + \int_{\frac{1}{2}}^1 \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx,$$

用比较判别法的极限形式, 对于 $\int_0^{\frac{1}{2}} \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx$, 由于 $\lim_{x \rightarrow 0^+} \frac{\frac{[\ln^2(1-x)]^{\frac{1}{m}}}{x^{\frac{1}{n}}}}{\frac{1}{x^{\frac{1}{n} - \frac{2}{m}}}} = 1$.

显然, 当 $0 < \frac{1}{n} - \frac{2}{m} < 1$, 则该反常积分收敛.

当 $\frac{1}{n} - \frac{2}{m} \leq 0$, $\lim_{x \rightarrow 0^+} \frac{[\ln^2(1-x)]^{\frac{1}{m}}}{x^{\frac{1}{n}}}$ 存在, 此时 $\int_0^{\frac{1}{2}} \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx$ 实际上不是反常积分, 故收敛.

敛.

故不论 m, n 是什么正整数, $\int_0^{\frac{1}{2}} \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx$ 总收敛. 对于 $\int_{\frac{1}{2}}^1 \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx$, 取

$0 < \delta < 1$, 不论 m, n 是什么正整数,

$$\lim_{x \rightarrow 1^-} \frac{\frac{[\ln^2(1-x)]^{\frac{1}{m}}}{x^{\frac{1}{n}}}}{\frac{1}{(1-x)^{\delta}}} = \lim_{x \rightarrow 1^-} \ln^2(1-x)^{\frac{1}{m}} (1-x)^{\delta} = 0,$$

所以 $\int_{\frac{1}{2}}^1 \frac{\sqrt[m]{\ln^2(1-x)}}{\sqrt[n]{x}} dx$ 收敛, 故选 (D).

(5) 【答案】 (B).

$$\text{【解析】 } \frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} = -\frac{F'_1\left(-\frac{y}{x^2}\right) + F'_2\left(-\frac{z}{x^2}\right)}{F'_2 \cdot \frac{1}{x}} = \frac{F'_1 \cdot \frac{y}{x} + F'_2 \cdot \frac{z}{x}}{F'_2} = \frac{yF'_1 + zF'_2}{xF'_2},$$

$$\frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z} = -\frac{F'_1 \cdot \frac{1}{x}}{F'_2 \cdot \frac{1}{x}} = -\frac{F'_1}{F'_2},$$

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{yF'_1 + zF'_2}{F'_2} - \frac{yF'_1}{F'_2} = \frac{F'_2 \cdot z}{F'_2} = z.$$

(6) 【答案】 (D).

$$\text{【解析】 } \sum_{i=1}^n \sum_{j=1}^n \frac{n}{(n+i)(n^2+j^2)} = \sum_{i=1}^n \frac{1}{n+i} \left(\sum_{j=1}^n \frac{n}{n^2+j^2} \right) = \left(\sum_{j=1}^n \frac{n}{n^2+j^2} \right) \left(\sum_{i=1}^n \frac{1}{n+i} \right)$$

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{n}{n^2+j^2} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \frac{1}{1+\left(\frac{j}{n}\right)^2} = \int_0^1 \frac{1}{1+y^2} dy,$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{n}{n+i} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{1+\left(\frac{i}{n}\right)} = \int_0^1 \frac{1}{1+x} dx,$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n \frac{n}{(n+i)(n^2+j^2)} = \lim_{n \rightarrow \infty} \left(\sum_{j=1}^n \frac{1}{n^2+j^2} \right) \left(\sum_{i=1}^n \frac{1}{n+i} \right)$$

$$= \left(\lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{n}{n^2+j^2} \right) \left(\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{n}{n+i} \right)$$

$$= \left(\int_0^1 \frac{1}{1+x} dx \right) \left(\int_0^1 \frac{1}{1+y^2} dy \right) = \int_0^1 dx \int_0^1 \frac{1}{(1+x)(1+y^2)} dy.$$

(7) 【答案】 (A).

【解析】 由于向量组 I 能由向量组 II 线性表示, 所以 $r(I) \leq r(II)$, 即

$$r(\alpha_1, \dots, \alpha_r) \leq r(\beta_1, \dots, \beta_s) \leq s$$

若向量组 I 线性无关, 则 $r(\alpha_1, \dots, \alpha_r) = r$, 所以 $r = r(\alpha_1, \dots, \alpha_r) \leq r(\beta_1, \dots, \beta_s) \leq s$, 即

$r \leq s$, 选 (A).

(8) 【答案】 (D).

【解析】：设 λ 为 A 的特征值, 由于 $A^2 + A = O$, 所以 $\lambda^2 + \lambda = 0$, 即 $(\lambda + 1)\lambda = 0$, 这样 A 的特征值只能为 -1 或 0 . 由于 A 为实对称矩阵, 故 A 可相似对角化, 即 $A \sim \Lambda$,

$$r(A) = r(\Lambda) = 3, \text{ 因此, } \Lambda = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \\ & & & 0 \end{pmatrix}, \text{ 即 } A \sim \Lambda = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \\ & & & 0 \end{pmatrix}.$$

二、填空题

(9) 【答案】 $y = C_1 e^{2x} + C_2 \cos x + C_3 \sin x$.

【解析】 该常系数线性齐次微分方程的特征方程为 $\lambda^3 - 2\lambda^2 + \lambda - 2 = 0$, 因式分解得

$$\lambda^2(\lambda - 2) + (\lambda - 2) = (\lambda - 2)(\lambda^2 + 1) = 0,$$

解得特征根为 $\lambda = 2, \lambda = \pm i$, 所以通解为 $y = C_1 e^{2x} + C_2 \cos x + C_3 \sin x$.

(10) 【答案】 $y = 2x$.

【解析】 因为 $\lim_{x \rightarrow \infty} \frac{2x^3}{x^2 + 1} = 2$, 所以函数存在斜渐近线, 又因为

$$\lim_{x \rightarrow \infty} \frac{2x^3}{x^2 + 1} - 2x = \lim_{x \rightarrow \infty} \frac{2x^3 - 2x^3 - 2x}{x^2 + 1} = 0, \text{ 所以斜渐近线方程为 } y = 2x.$$

(11) 【答案】 $-2^n \cdot (n-1)!$.

【解析】 由高阶导数公式可知 $\ln^{(n)}(1+x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n}$,

$$\text{所以 } \ln^{(n)}(1-2x) = (-1)^{n-1} \frac{(n-1)!}{(1-2x)^n} \cdot (-2)^n = -2^n \frac{(n-1)!}{(1-2x)^n},$$

$$\text{即 } y^{(n)}(0) = -2^n \frac{(n-1)!}{(1-2 \cdot 0)^n} = -2^n (n-1)!.$$

(12) 【答案】 $\sqrt{2}(e^\pi - 1)$.

【解析】 因为 $0 \leq \theta \leq \pi$, 所以对数螺线 $r = e^\theta$ 的极坐标弧长公式为

$$\int_0^\pi \sqrt{(e^\theta)^2 + (e^\theta)^2} d\theta = \int_0^\pi \sqrt{2} e^\theta d\theta = \sqrt{2}(e^\pi - 1).$$

(13) 【答案】 3 cm/s .

【解析】设 $l = x(t)$, $w = y(t)$, 由题意知, 在 $t = t_0$ 时刻 $x(t_0) = 12$, $y(t_0) = 5$, 且 $x'(t_0) = 2$,

$y'(t_0) = 3$, 设该对角线长为 $S(t)$, 则 $S(t) = \sqrt{x^2(t) + y^2(t)}$, 所以

$$S'(t) = \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x^2(t) + y^2(t)}}.$$

所以
$$S'(t_0) = \frac{x(t_0)x'(t_0) + y(t_0)y'(t_0)}{\sqrt{x^2(t_0) + y^2(t_0)}} = \frac{12 \cdot 2 + 5 \cdot 3}{\sqrt{12^2 + 5^2}} = 3.$$

(14) 【答案】3.

【解析】由于 $A(A^{-1} + B)B^{-1} = (E + AB)B^{-1} = B^{-1} + A$, 所以

$$|A + B^{-1}| = |A(A^{-1} + B)B^{-1}| = |A||A^{-1} + B||B^{-1}|$$

因为 $|B| = 2$, 所以 $|B^{-1}| = |B|^{-1} = \frac{1}{2}$, 因此

$$|A + B^{-1}| = |A||A^{-1} + B||B^{-1}| = 3 \times 2 \times \frac{1}{2} = 3.$$

三、解答题

(15) 【解析】因为 $f(x) = \int_1^{x^2} (x^2 - t)e^{-t^2} dt = x^2 \int_1^{x^2} e^{-t^2} dt - \int_1^{x^2} te^{-t^2} dt$,

所以 $f'(x) = 2x \int_1^{x^2} e^{-t^2} dt + 2x^3 e^{-x^4} - 2x^3 e^{-x^4} = 2x \int_1^{x^2} e^{-t^2} dt$, 令 $f'(x) = 0$, 则

$x = 0, x = \pm 1$.

又 $f''(x) = 2 \int_1^{x^2} e^{-t^2} dt + 4x^2 e^{-x^4}$, 则 $f''(0) = 2 \int_1^0 e^{-t^2} dt < 0$, 所以

$$f(0) = \int_1^0 (0 - t)e^{-t^2} dt = -\frac{1}{2}e^{-t^2} \Big|_0^1 = \frac{1}{2}(1 - e^{-1})$$

是极大值.

而 $f''(\pm 1) = 4e^{-1} > 0$, 所以 $f(\pm 1) = 0$ 为极小值.

又因为当 $x \geq 1$ 时, $f'(x) > 0$; $0 \leq x < 1$ 时, $f'(x) < 0$; $-1 \leq x < 0$ 时, $f'(x) > 0$;

$x < -1$ 时, $f'(x) < 0$, 所以 $f(x)$ 的单调递减区间为 $(-\infty, -1) \cup (0, 1)$, $f(x)$ 的单调递增区

间为 $(-1, 0) \cup (1, +\infty)$.

(16) 【解析】(I) 当 $0 < x < 1$ 时 $0 < \ln(1+x) < x$, 故 $[\ln(1+t)]^n < t^n$, 所以

$$|\ln t| [\ln(1+t)]^n < |\ln t| t^n,$$

则

$$\int_0^1 |\ln t| [\ln(1+t)]^n dt < \int_0^1 |\ln t| t^n dt \quad (n=1, 2, \dots).$$

$$(II) \int_0^1 |\ln t| t^n dt = -\int_0^1 \ln t \cdot t^n dt = -\frac{1}{n+1} \int_0^1 \ln t d(t^{n+1}) = \frac{1}{(n+1)^2}, \text{ 故由}$$

$$0 < u_n < \int_0^1 |\ln t| t^n dt = \frac{1}{(n+1)^2},$$

根据夹逼定理得 $0 \leq \lim_{n \rightarrow \infty} u_n \leq \lim_{n \rightarrow \infty} \frac{1}{(n+1)^2} = 0$, 所以 $\lim_{n \rightarrow \infty} u_n = 0$.

(17) 【解析】根据题意得

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\psi'(t)}{2t+2}, \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{\psi'(t)}{2t+2}\right)}{\frac{dx}{dt}} = \frac{\psi''(t)(2t+2) - 2\psi'(t)}{(2t+2)^2} = \frac{3}{4(1+t)}$$

即 $\psi''(t)(2t+2) - 2\psi'(t) = 6(t+1)^2$, 整理有 $\psi''(t)(t+1) - \psi'(t) = 3(t+1)^2$, 解

$$\begin{cases} \psi''(t) - \frac{\psi'(t)}{t+1} = 3(t+1) \\ \psi(1) = \frac{5}{2}, \psi'(1) = 6 \end{cases}, \text{ 令 } y = \psi'(t), \text{ 即 } y' - \frac{1}{1+t}y = 3(1+t).$$

所以 $y = e^{\int \frac{1}{1+t} dt} \left(\int 3(1+t) e^{-\int \frac{1}{1+t} dt} dt + C \right) = (1+t)(3t+C)$, $t > -1$. 因为 $y(1) = \psi'(1) = 6$,

所以 $C = 0$, 故 $y = 3t(t+1)$, 即 $\psi'(t) = 3t(t+1)$,

故 $\psi(t) = \int 3t(t+1) dt = \frac{3}{2}t^2 + t^3 + C_1$.

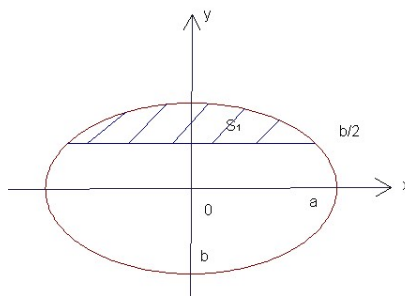
又由 $\psi(1) = \frac{5}{2}$, 所以 $C_1 = 0$, 故 $\psi(t) = \frac{3}{2}t^2 + t^3, (t > -1)$.

(18) 【解析】油罐放平, 截面如图建立坐标系之后, 边界椭圆的方程为:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

阴影部分的面积

$$S = \int_{-\frac{b}{2}}^{\frac{b}{2}} 2x dy = \frac{2a}{b} \int_{-\frac{b}{2}}^{\frac{b}{2}} \sqrt{b^2 - y^2} dy$$



令 $y = b \sin t$, $y = -b$ 时 $t = -\frac{\pi}{2}$; $y = \frac{b}{2}$ 时 $t = \frac{\pi}{6}$.

$$S = 2ab \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \cos^2 t dt = 2ab \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(\frac{1}{2} + \frac{1}{2} \cos 2t \right) dt = \left(\frac{2}{3} \pi + \frac{\sqrt{3}}{4} \right) ab$$

所以油的质量 $m = \left(\frac{2}{3} \pi + \frac{\sqrt{3}}{4} \right) abl \rho$.

(19) 【解析】由复合函数链式法则得

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} \cdot a + b \cdot \frac{\partial u}{\partial \eta},$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right) = \frac{\partial^2 u}{\partial \xi^2} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{\partial \eta}{\partial x} \\ &= \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right) = \frac{\partial^2 u}{\partial \xi^2} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{\partial \eta}{\partial y} \\ &= a \frac{\partial^2 u}{\partial \xi^2} + b \frac{\partial^2 u}{\partial \eta^2} + (a+b) \frac{\partial^2 u}{\partial \xi \partial \eta}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y} \left(a \frac{\partial u}{\partial \xi} + b \frac{\partial u}{\partial \eta} \right) = a \left(a \frac{\partial^2 u}{\partial \xi^2} + b \frac{\partial^2 u}{\partial \xi \partial \eta} \right) + b \left(a \frac{\partial^2 u}{\partial \eta^2} + a \frac{\partial^2 u}{\partial \xi \partial \eta} \right) \\ &= a^2 \frac{\partial^2 u}{\partial \xi^2} + b^2 \frac{\partial^2 u}{\partial \eta^2} + 2ab \frac{\partial^2 u}{\partial \xi \partial \eta}, \end{aligned}$$

$$\text{故 } 4 \frac{\partial^2 u}{\partial x^2} + 12 \frac{\partial^2 u}{\partial x \partial y} + 5 \frac{\partial^2 u}{\partial y^2}$$

$$= (5a^2 + 12a + 4) \frac{\partial^2 u}{\partial \xi^2} + (5b^2 + 12b + 4) \frac{\partial^2 u}{\partial \eta^2} + [12(a+b) + 10ab + 8] \frac{\partial^2 u}{\partial \xi \partial \eta} = 0,$$

$$\text{所以 } \begin{cases} 5a^2 + 12a + 4 = 0 \\ 5b^2 + 12b + 4 = 0 \\ 12(a+b) + 10ab + 8 \neq 0 \end{cases},$$

则 $a = -\frac{2}{5}$ 或 -2 , $b = -\frac{2}{5}$ 或 -2 . 又因为当 (a, b) 为 $(-2, -2), (-\frac{2}{5}, -\frac{2}{5})$ 时方程 (3) 不满足, 所以当 (a, b) 为 $(-\frac{2}{5}, -2), (-2, -\frac{2}{5})$ 满足题意.

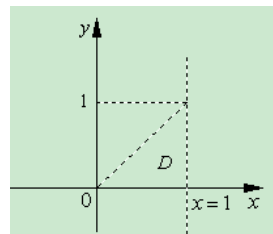
(20) 【解析】 $I = \iint_D r^2 \sin \theta \sqrt{1-r^2} \cos 2\theta dr d\theta$

$$= \iint_D r \sin \theta \sqrt{1-r^2 (\cos^2 \theta - \sin^2 \theta)} \cdot r dr d\theta$$

$$= \iint_D y \sqrt{1-x^2+y^2} dx dy$$

$$= \int_0^1 dx \int_0^x y \sqrt{1-x^2+y^2} dy = \int_0^1 \frac{1}{3} \left[1 - (1-x^2)^{\frac{3}{2}} \right] dx$$

$$= \int_0^1 \frac{1}{3} dx - \frac{1}{3} \int_0^1 (1-x^2)^{\frac{3}{2}} dx = \frac{1}{3} - \int_0^{\frac{\pi}{2}} \cos 4\theta d\theta = \frac{1}{3} - \frac{3}{16} \pi.$$



(21) 【解析】 令 $F(x) = f(x) - \frac{1}{3}x^3$, 对于 $F(x)$ 在 $\left[0, \frac{1}{2}\right]$ 上利用拉格朗日中值定理, 得存

在 $\xi \in \left(0, \frac{1}{2}\right)$, 使得

$$F\left(\frac{1}{2}\right) - F(0) = \frac{1}{2} F'(\xi).$$

对于 $F(x)$ 在 $\left[\frac{1}{2}, 1\right]$ 上利用拉格朗日中值定理, 得存在 $\eta \in \left(\frac{1}{2}, 1\right)$, 使得

$$F(1) - F\left(\frac{1}{2}\right) = \frac{1}{2} F'(\eta),$$

两式相加得

$$f'(\xi) + f'(\eta) = \xi^2 + \eta^2.$$

所以存在 $\xi \in \left(0, \frac{1}{2}\right), \eta \in \left(\frac{1}{2}, 1\right)$, 使 $f'(\xi) + f'(\eta) = \xi^2 + \eta^2$.

(22) 【解析】 因为方程组有两个不同的解, 所以可以判断方程组增广矩阵的秩小于 3, 进而可以通过秩的关系求解方程组中未知参数, 有以下两种方法.

方法 1: (I) 已知 $Ax = b$ 有 2 个不同的解, 故 $r(A) = r(\bar{A}) < 3$, 对增广矩阵进行初等行变换, 得

$$\begin{aligned}\bar{A} &= \left(\begin{array}{ccc|c} \lambda & 1 & 1 & a \\ 0 & \lambda-1 & 0 & 1 \\ 1 & 1 & \lambda & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 0 & 1 \\ \lambda & 1 & 1 & a \end{array} \right) \\ &\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 0 & 1 \\ 0 & 1-\lambda & 1-\lambda^2 & a-\lambda \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 0 & 1 \\ 0 & 0 & 1-\lambda^2 & a-\lambda+1 \end{array} \right)\end{aligned}$$

当 $\lambda=1$ 时, $\bar{A} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$, 此时, $r(A) \neq r(\bar{A})$, 故 $Ax=b$ 无解 (舍去).

当 $\lambda=-1$ 时, $\bar{A} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & a+2 \end{array} \right)$, 由于 $r(A)=r(\bar{A})<3$, 所以 $a=-2$, 故 $\lambda=-1$, $a=-2$.

方法 2: 已知 $Ax=b$ 有 2 个不同的解, 故 $r(A)=r(\bar{A})<3$, 因此 $|A|=0$, 即

$$|A| = \begin{vmatrix} \lambda & 1 & 1 \\ 0 & \lambda-1 & 0 \\ 1 & 1 & \lambda \end{vmatrix} = (\lambda-1)^2(\lambda+1) = 0,$$

知 $\lambda=1$ 或 -1 .

当 $\lambda=1$ 时, $r(A)=1 \neq r(\bar{A})=2$, 此时, $Ax=b$ 无解, 因此 $\lambda=-1$. 由 $r(A)=r(\bar{A})$, 得 $a=-2$.

(II) 对增广矩阵做初等行变换

$$\bar{A} = \left(\begin{array}{ccc|c} -1 & 1 & 1 & -2 \\ 0 & -2 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array} \right)$$

可知原方程组等价于 $\begin{cases} x_1 - x_3 = \frac{3}{2} \\ x_2 = -\frac{1}{2} \end{cases}$, 写成向量的形式, 即 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$.

因此 $Ax=b$ 的通解为 $x=k\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+\begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$, 其中 k 为任意常数.

(23) 【解析】由于 $A=\begin{pmatrix} 0 & -1 & 4 \\ -1 & 3 & a \\ 4 & a & 0 \end{pmatrix}$, 存在正交矩阵 Q , 使得 $Q^T A Q$ 为对角阵, 且 Q 的第一

列为 $\frac{1}{\sqrt{6}}(1, 2, 1)^T$, 故 A 对应于 λ_1 的特征向量为 $\xi_1 = \frac{1}{\sqrt{6}}(1, 2, 1)^T$.

根据特征值和特征向量的定义, 有 $A\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} = \lambda_1 \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$, 即

$$\begin{pmatrix} 0 & -1 & 4 \\ -1 & 3 & a \\ 4 & a & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \text{ 由此可得 } a = -1, \lambda_1 = 2. \text{ 故 } A = \begin{pmatrix} 0 & -1 & 4 \\ -1 & 3 & -1 \\ 4 & -1 & 0 \end{pmatrix}.$$

$$\text{由 } |\lambda E - A| = \begin{vmatrix} \lambda & 1 & -4 \\ 1 & \lambda - 3 & 1 \\ -4 & 1 & \lambda \end{vmatrix} = (\lambda + 4)(\lambda - 2)(\lambda - 5) = 0,$$

可得 A 的特征值为 $\lambda_1 = 2, \lambda_2 = -4, \lambda_3 = 5$.

$$\text{由 } (\lambda_2 E - A)x = 0, \text{ 即 } \begin{pmatrix} -4 & 1 & -4 \\ 1 & -7 & 1 \\ -4 & 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \text{ 可解得对应于 } \lambda_2 = -4 \text{ 的线性无关的}$$

特征向量为 $\xi_2 = (-1, 0, 1)^T$.

$$\text{由 } (\lambda_3 E - A)x = 0, \text{ 即 } \begin{pmatrix} 5 & 1 & -4 \\ 1 & 2 & 1 \\ -4 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \text{ 可解得对应于 } \lambda_3 = 5 \text{ 的特征向量为}$$

$$\xi_3 = (1, -1, 1)^T.$$

由于 A 为实对称矩阵, ξ_1, ξ_2, ξ_3 为对应于不同特征值的特征向量, 所以 ξ_1, ξ_2, ξ_3 相互正

交, 只需单位化:

$$\eta_1 = \frac{\xi_1}{\|\xi_1\|} = \frac{1}{\sqrt{6}}(1, 2, 1)^T, \eta_2 = \frac{\xi_2}{\|\xi_2\|} = \frac{1}{\sqrt{2}}(-1, 0, 1)^T, \eta_3 = \frac{\xi_3}{\|\xi_3\|} = \frac{1}{\sqrt{3}}(1, -1, 1)^T,$$

$$\text{取 } Q = (\eta_1, \eta_2, \eta_3) = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \text{ 则 } Q^T A Q = \Lambda = \begin{pmatrix} 2 & & \\ & -4 & \\ & & 5 \end{pmatrix}.$$