

2008年(数一)真题答案解析

一、选择题

(1) B

解 $f'(x) = \ln(2+x^2) \cdot 2x = 2x \ln(2+x^2)$,

又因为 $\ln(2+x^2) \neq 0$,

所以 $f'(x)$ 只有一个零点.

(2) A

解 由 $f'_x = \frac{\frac{1}{y}}{1+\frac{x^2}{y^2}} = \frac{\frac{1}{y}}{\frac{x^2+y^2}{y^2}} = \frac{y}{x^2+y^2}$, 得 $f'_x(0,1) = \frac{1}{1} = 1$.

由 $f'_y = \frac{\frac{-x}{y^2}}{1+\frac{x^2}{y^2}} = \frac{-x}{x^2+y^2}$, 得 $f'_y(0,1) = 0$.

所以 $\text{grad}f(0,1) = 1 \times i + 0 \times j = i$.

(3) D

解 由 $y = C_1 e^x + C_2 \cos 2x + C_3 \sin 2x$ 可知其特征根为 $\lambda_1 = 1, \lambda_{2,3} = \pm 2i$.

故对应的特征方程为

$$(\lambda - 1)(\lambda + 2i)(\lambda - 2i) = (\lambda - 1)(\lambda^2 + 4),$$

即 $\lambda^3 - \lambda^2 + 4\lambda - 4 = 0$,

所以所求微分方程为

$$y''' - y'' + 4y' - 4y = 0.$$

(4) B

解 若 $\{x_n\}$ 单调, 则由 $f(x)$ 在 $(-\infty, +\infty)$ 内单调有界知, $\{f(x_n)\}$ 单调有界, 因此 $\{f(x_n)\}$ 收敛.

(5) C

解 $(E - A)(E + A + A^2) = E - A^3 = E$,

$(E + A)(E - A + A^2) = E + A^3 = E$,

所以 $E - A, E + A$ 均可逆.

(6) B

解 此二次曲面为旋转双叶双曲面, 此曲面的标准方程为 $\frac{x^2}{a^2} - \frac{y^2 + z^2}{c^2} = 1$, 所以 A 的正特征值个数为 1.

(7) A

解 设 Z 的分布函数为 $F_Z(x)$, 则

$$F_Z(x) = P(Z \leq x) = P\{\max\{X, Y\} \leq x\} = P\{X \leq x, Y \leq x\}.$$

由于 X 与 Y 独立同分布, 于是有

$$F_Z(x) = P(X \leq x)P(Y \leq x) = F^2(x).$$

(8) D

解 用排除法.

设 $Y = aX + b$, 由 $\rho_{XY} = 1$ 知 X, Y 相关, 得 $a > 0$, 排除 A、C;

由 $X \sim N(0, 1), Y \sim N(1, 4)$ 得

$$EX = 0, EY = 1,$$

$$E(Y) = E(aX + b) = aEX + b,$$

$$1 = a \times 0 + b, b = 1. \text{ 排除 B. 故应选 D.}$$

二、填空题

(9) $\frac{1}{x}$

$$\text{解 由 } \frac{dy}{dx} = \frac{-y}{x}, \text{ 即 } \frac{dy}{-y} = \frac{dx}{x},$$

$$\text{积分得 } -\ln|y| = \ln|x| + C_1,$$

$$\text{所以 } \frac{1}{|y|} = |x| + C.$$

$$\text{又 } y(1) = 1, \text{ 所以 } y = \frac{1}{x}.$$

(10) $y = x + 1$

解 设 $F(x, y) = \sin(xy) + \ln(y - x) - x$, 斜率

$$k = -\frac{F_x}{F_y} = -\frac{y \cos(xy) + \frac{-1}{y-x} - 1}{x \cos(xy) + \frac{1}{y-x}},$$

在 $(0, 1)$ 处, $k = 1$, 所以切线方程为 $y - 1 = x$, 即 $y = x + 1$.

(11) $(1, 5]$

解 由题意知 $\sum_{n=0}^{\infty} a_n (x+2)^n$ 的收敛域为 $(-4, 0]$, 则 $\sum_{n=0}^{\infty} a_n x^n$ 的收敛域为 $(-2, 2]$.

所以 $\sum_{n=0}^{\infty} a_n (x-3)^n$ 的收敛域为 $(1, 5]$.

(12) 4π

$$\begin{aligned} \text{解 } & \iint_{\Sigma} xy dy dz + x dz dx + x^2 dx dy \\ &= \oint_{\Sigma+D} xy dy dz + x dz dx + x^2 dx dy + \iint_{D_{\perp}} x^2 dx dy \\ &= \iiint_{\Omega} y dx dy dz + \iint_{D_{\perp}} x^2 dx dy \\ &= 0 + \frac{1}{2} \iint_{D_{\perp}} (x^2 + y^2) dx dy \end{aligned}$$

$$= \frac{1}{2} \int_0^{2\pi} d\theta \int_0^2 r^2 r dr$$

$$= 4\pi.$$

(13) 1

解 $A[\alpha_1, \alpha_2] = [A\alpha_1, A\alpha_2] = [0, 2\alpha_1 + \alpha_2] = [\alpha_1, \alpha_2] \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}.$

记 $P = [\alpha_1, \alpha_2]$, P 可逆, 故

$$P^{-1}AP = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = B.$$

A 有 B 相同的特征值

$$|\lambda E - B| = \begin{vmatrix} \lambda & -2 \\ 0 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1),$$

$$\lambda_{1,2} = 0, 1,$$

所以非零的特征值为 1.

(14) $\frac{1}{2e}$

解 因为 $DX = EX^2 - (EX)^2$, 所以 $EX^2 = 2$, X 服从参数为 1 的泊松分布,

$$\text{所以 } P\{X=2\} = \frac{1}{2}e^{-1} = \frac{1}{2e}.$$

三、解答题

(15) 解 $\lim_{x \rightarrow 0} \frac{[\sin x - \sin(\sin x)] \sin x}{x^4}$

$$= \lim_{x \rightarrow 0} \frac{\sin x - \sin(\sin x)}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{\cos x - \cos(\sin x) \cos x}{3x^2}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos(\sin x)}{3x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{2} \sin^2 x}{3x^2}$$

$$= \frac{1}{6}.$$

(16) 解法一 $\int_L \sin 2x dx + 2(x^2 - 1)y dy$

$$= \int_0^\pi [\sin 2x + 2(x^2 - 1) \sin x \cdot \cos x] dx$$

$$= \int_0^\pi x^2 \sin 2x dx$$

$$= -\frac{x^2}{2} \cos 2x \Big|_0^\pi + \int_0^\pi x \cos 2x dx$$

$$\begin{aligned}
&= -\frac{\pi^2}{2} + \frac{x}{2} \sin 2x \Big|_0^\pi - \frac{1}{2} \int_0^\pi \sin 2x \, dx \\
&= -\frac{\pi^2}{2}.
\end{aligned}$$

解法二 取 L_1 为 x 轴上从点 $(\pi, 0)$ 到点 $(0, 0)$ 的一段, D 是由 L 与 L_1 围成的区域.

$$\begin{aligned}
&\int_L \sin 2x \, dx + 2(x^2 - 1)y \, dy \\
&= \int_{L+L_1} \sin 2x \, dx + 2(x^2 - 1)y \, dy - \int_{L_1} \sin 2x \, dx + 2(x^2 - 1)y \, dy \\
&= -\iint_D 4xy \, dx \, dy - \int_0^\pi \sin 2x \, dx \\
&= -\int_0^\pi dx \int_0^{\sin x} 4xy \, dy - \frac{1}{2} \cos 2x \Big|_0^\pi \\
&= -\int_0^\pi 2x \sin^2 x \, dx \\
&= -\int_0^\pi x(1 - \cos 2x) \, dx \\
&= -\frac{x^2}{2} \Big|_0^\pi + \frac{x}{2} \sin 2x \Big|_0^\pi - \frac{1}{2} \int_0^\pi \sin 2x \, dx \\
&= -\frac{\pi^2}{2}.
\end{aligned}$$

(17) **解** 点 (x, y, z) 到 xOy 面的距离为 $|z|$, 故求 C 上距离 xOy 面最远点和最近点的坐标, 等价于求函数 $H = z^2$ 在条件 $x^2 + y^2 - 2z^2 = 0$ 与 $x + y + 3z = 5$ 下的最大值点和最小值点.

令 $L(x, y, z, \lambda, \mu) = z^2 + \lambda(x^2 + y^2 - 2z^2) + \mu(x + y + 3z - 5)$,

由

$$\begin{cases} L'_x = 2\lambda x + \mu = 0, \\ L'_y = 2\lambda y + \mu = 0, \\ L'_z = 2z - 4\lambda z + 3\mu = 0, \\ x^2 + y^2 - 2z^2 = 0, \\ x + y + 3z = 5, \end{cases}$$

得 $x = y$, 从而

$$\begin{cases} 2x^2 - 2z^2 = 0, \\ 2x + 3z = 5, \end{cases}$$

解得

$$\begin{cases} x = -5, \\ y = -5, \\ z = 5, \end{cases} \quad \text{或} \quad \begin{cases} x = 1, \\ y = 1, \\ z = 1. \end{cases}$$

根据几何意义, 曲线 C 上存在距离 xOy 面最远的点和最近的点, 故所求点依次为 $(-5, -5, 5)$ 和 $(1, 1, 1)$.

(18) (I) 证 对任意的 x , 由于 f 是连续函数, 所以

$$\begin{aligned} & \lim_{\Delta x \rightarrow 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\int_0^{x+\Delta x} f(t) dt - \int_0^x f(t) dt}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\int_x^{x+\Delta x} f(t) dt}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(\xi) \Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} f(\xi), \end{aligned}$$

其中 ξ 介于 x 与 $x+\Delta x$ 之间.

由 $\lim_{\Delta x \rightarrow 0} f(\xi) = f(x)$, 可知函数 $F(x)$ 在 x 处可导, 且 $F'(x) = f(x)$.

(II) 证法一 要证明 $G(x)$ 以 2 为周期, 即要证明对任意的 x , 都有 $G(x+2) = G(x)$.

记 $H(x) = G(x+2) - G(x)$, 则

$$\begin{aligned} H'(x) &= \left(2 \int_0^{x+2} f(t) dt - (x+2) \int_0^2 f(t) dt \right)' - \left(2 \int_0^x f(t) dt - x \int_0^2 f(t) dt \right)' \\ &= 2f(x+2) - \int_0^2 f(t) dt - 2f(x) + \int_0^2 f(t) dt \\ &= 0, \end{aligned}$$

又因为

$$H(0) = G(2) - G(0) = \left(2 \int_0^2 f(t) dt - 2 \int_0^2 f(t) dt \right) - 0 = 0,$$

所以

$$H(x) = 0, \text{ 即 } G(x+2) = G(x).$$

证法二 由于 f 是以 2 为周期的连续函数, 所以对任意的 x , 有

$$\begin{aligned} G(x+2) - G(x) &= 2 \int_0^{x+2} f(t) dt - (x+2) \int_0^2 f(t) dt - 2 \int_0^x f(t) dt + x \int_0^2 f(t) dt \\ &= 2 \left[\int_0^2 f(t) dt + \int_2^{x+2} f(t) dt - \int_0^2 f(t) dt - \int_0^x f(t) dt \right] \\ &= 2 \left[\int_0^x f(u+2) du - \int_0^x f(t) dt \right] \\ &= 2 \int_0^x [f(t+2) - f(t)] dt \\ &= 0, \end{aligned}$$

即 $G(x)$ 是以 2 为周期的周期函数.

(19) 解 由于

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi (1-x^2) dx = 2 - \frac{2\pi^2}{3}, \\ a_n &= \frac{2}{\pi} \int_0^\pi (1-x^2) \cos nx dx \end{aligned}$$

$$= \frac{4}{n^2}(-1)^{n+1}, n=1, 2, \dots,$$

所以

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= 1 - \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx, \quad 0 \leq x \leq \pi. \end{aligned}$$

令 $x=0$, 有

$$f(0) = 1 - \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2},$$

又 $f(0)=1$, 所以

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

$$\begin{aligned} (20) \text{ 证 } (I) \quad r(A) &= r(\alpha\alpha^T + \beta\beta^T) \\ &\leq r(\alpha\alpha^T) + r(\beta\beta^T) \\ &\leq r(\alpha) + r(\beta) \\ &\leq 2. \end{aligned}$$

(II) 由于 α, β 线性相关, 不妨设 $\alpha = k\beta$, 于是

$$\begin{aligned} r(A) &= r(\alpha\alpha^T + \beta\beta^T) \\ &= r((1+k^2)\beta\beta^T) \\ &\leq r(\beta) \leq 1 < 2. \end{aligned}$$

(21) (I) 证 记

$$D_n = |A| = \begin{vmatrix} 2a & 1 & & & \\ a^2 & 2a & 1 & & \\ & a^2 & 2a & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & a^2 & 2a & 1 \\ & & & & a^2 & 2a \end{vmatrix}_n,$$

以下用数学归纳法证明 $D_n = (n+1)a^n$.

当 $n=1$ 时, $D_1 = 2a$, 结论成立.

$$\text{当 } n=2 \text{ 时, } D_2 = \begin{vmatrix} 2a & 1 \\ a^2 & 2a \end{vmatrix} = 3a^2, \text{ 结论成立.}$$

假设结论对小于 n 的情况成立. 将 D_n 按第 1 行展开, 得

$$D_n = 2aD_{n-1} - \begin{vmatrix} a^2 & 1 & & & \\ 0 & 2a & 1 & & \\ & a^2 & 2a & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & a^2 & 2a & 1 \\ & & & & a^2 & 2a \end{vmatrix}_{n-1}$$

$$\begin{aligned}
&= 2aD_{n-1} - a^2D_{n-2} \\
&= 2ana^{n-1} - a^2(n-1)a^{n-2} \\
&= (n+1)a^n,
\end{aligned}$$

故 $|\mathbf{A}| = (n+1)a^n$.

(II) 解 当 $a \neq 0$ 时, 方程组系数行列式 $D_n \neq 0$, 故方程组有唯一解. 由克莱姆法则, 将 D_n 的第 1 列换成 b , 得行列式为

$$\begin{vmatrix} 1 & 1 & & & \\ 0 & 2a & 1 & & \\ & a^2 & 2a & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & a^2 & 2a & 1 \\ & & & & a^2 & 2a \end{vmatrix}_n = \begin{vmatrix} 2a & 1 & & & \\ a^2 & 2a & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & a^2 & 2a & 1 \\ & & & a^2 & 2a \end{vmatrix}_{n-1} = D_{n-1} = na^{n-1},$$

所以 $x_1 = \frac{D_{n-1}}{D_n} = \frac{n}{(n+1)a}$.

(III) 解 当 $a = 0$ 时, 方程组为

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

此时方程组系数矩阵的秩和增广矩阵的秩均为 $n-1$, 所以方程组有无穷多解, 其通解为

$$\mathbf{x} = (0, 1, 0, \dots, 0)^T + k(1, 0, 0, \dots, 0)^T,$$

其中 k 为任意常数.

$$\begin{aligned}
(22) \text{ 解 } (I) P\left\{Z \leq \frac{1}{2} \mid X=0\right\} &= \frac{P\left\{X=0, Z \leq \frac{1}{2}\right\}}{P\{X=0\}} \\
&= \frac{P\left\{X=0, Y \leq \frac{1}{2}\right\}}{P\{X=0\}} \\
&= P\left\{Y \leq \frac{1}{2}\right\} = \frac{1}{2}.
\end{aligned}$$

$$\begin{aligned}
(II) F_Z(z) &= P\{Z \leq z\} = P\{X+Y \leq z\} \\
&= P\{X+Y \leq z, X=-1\} + P\{X+Y \leq z, X=0\} + P\{X+Y \leq z, X=1\} \\
&= P\{Y \leq z+1, X=-1\} + P\{Y \leq z, X=0\} + P\{Y \leq z-1, X=1\} \\
&= P\{Y \leq z+1\}P\{X=-1\} + P\{Y \leq z\}P\{X=0\} + P\{Y \leq z-1\}P\{X=1\} \\
&= \frac{1}{3}[P\{Y \leq z+1\} + P\{Y \leq z\} + P\{Y \leq z-1\}] \\
&= \frac{1}{3}[F_Y(z+1) + F_Y(z) + F_Y(z-1)],
\end{aligned}$$

$$\begin{aligned}
 f_Z(z) &= F'_Z(z) \\
 &= \frac{1}{3} [f_Y(z+1) + f_Y(z) + f_Y(z-1)] \\
 &= \begin{cases} \frac{1}{3}, & -1 \leq z < 2, \\ 0, & \text{其他.} \end{cases}
 \end{aligned}$$

(23) (I) 证 因为

$$\begin{aligned}
 ET &= E\left(\bar{X}^2 - \frac{1}{n}S^2\right) \\
 &= E\bar{X}^2 - \frac{1}{n}ES^2 \\
 &= (E\bar{X})^2 + D\bar{X} - \frac{1}{n}ES^2 \\
 &= \mu^2 + \frac{\sigma^2}{n} - \frac{\sigma^2}{n} \\
 &= \mu^2, \text{ 所以 } T \text{ 是 } \mu^2 \text{ 的无偏估计量.}
 \end{aligned}$$

(II) 解 当 $\mu=0, \sigma=1$ 时, 有

$$\begin{aligned}
 DT &= D\left(\bar{X}^2 - \frac{1}{n}S^2\right) \text{ (注意 } \bar{X} \text{ 与 } S^2 \text{ 独立)} \\
 &= D\bar{X}^2 + \frac{1}{n^2}DS^2 \\
 &= \frac{1}{n^2}D(\sqrt{n}\bar{X})^2 + \frac{1}{n^2} \cdot \frac{1}{(n-1)^2}D[(n-1)S^2] \\
 &= \frac{1}{n^2} \cdot 2 + \frac{1}{n^2} \cdot \frac{1}{(n-1)^2} \cdot 2(n-1) \\
 &= \frac{2}{n^2} \cdot \left(1 + \frac{1}{n-1}\right) \\
 &= \frac{2}{n(n-1)}.
 \end{aligned}$$